Algebro-geometric approaches to the tensor eigenproblem
joint work with F. Galuppi, F. Gesmundo, P. Santarsiero, and T. Seynnaeve
(still working in progress)

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What is algebraic geometry?

- **Algebraic geometry** is the study of geometric objects determined by systems of polynomials equations called *varieties*.

- If $X$ is a variety, then one can ask the following questions:
  
  **Questions**
  
  (a) How many irreducible components does $X$ have?
  
  (b) How big are the components? What are their degrees?
  
  (c) Is each component non-singular?

- To answer (a) and (b), compute the “primary decomposition” of the ideal of $X$ and the “Hilbert polynomial” of its each primary component.
What is algebraic geometry? (cont’d)

• One can also consider a continuous family of varieties with prescribed invariants (such as dimension and degree) and ask:

**Question**
What are the limits of varieties in the family?

**Example**
Let $A_t := \begin{bmatrix} 2t + 3 & 1 \\ 0 & t + 3 \end{bmatrix}$. The characteristic polynomial of $A_t$ is

$$x^2 + (-3t - 6)x + 2t^2 - 9t + 9$$

and its discriminant is $t^2$.

• The theory of “schemes” is a useful resource to answer this question.
Eigenvectors of matrices

**Definition**

Let $A$ be an $n \times n$ matrix whose entries are from $\mathbb{C}$. A non-zero vector $v \in \mathbb{C}^n$ is called an *eigenvector* of $A$ if $\exists \lambda \in \mathbb{C}$ such that

$$Av = \lambda v.$$ 

**Remark**

The following are all equivalent:

1. $v \in \mathbb{C}^n \setminus \{0\}$ is an eigenvector of $A$.
2. $Av$ and $v$ are linearly dependent.
3. The $n \times 2$ matrix $(Av \mid v)$ obtained by concatenating $Av$ and $v$ side by side has rank 1.
4. The $2 \times 2$ minors of $(Av \mid v)$ are all zero.
Eigenvectors of matrices (cont’d)

• $R := \mathbb{C}[x_1, x_2, \ldots, x_n]$.  
• $\mathbf{x} := (x_1, x_2, \ldots, x_n)^T$.  
• $I_A \subseteq R$: ideal generated by the $2 \times 2$ minors of $(Ax \mid x)$.  

Remark

A vector $\mathbf{v} \in \mathbb{C}^n \setminus \{0\}$ is an eigenvector of $A$ iff $\alpha \mathbf{v}$ is an eigenvector of $A$ for every $\alpha \in \mathbb{C} := \mathbb{C}^\times \setminus \{0\}$.

Definition

(1) If $\mathbf{v} \in \mathbb{C}^n$ is an eigenvector of $A$, then we call $[\mathbf{v}]$ the eigenpoint of $A$.
(2) We call the scheme $Z_A$ defined by $I_A$ the eigenscheme of $A$.
(2) The reduced scheme associated to $Z_A$ is called the eigenvariety of $A$ and denoted by $V_A$.  

Example

If \( A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \), then \( I_A \) is generated by

\[
\det(Ax \mid x) = \begin{vmatrix} \lambda x_1 + x_2 & x_1 \\ \lambda x_2 & x_2 \end{vmatrix} = x_2^2
\]

Therefore, \( Z_A \) is a zero-dimensional subscheme of \( \mathbb{P}^1 \) with length 2 supported at \([(1,0)^T]\). In particular, \( Z_A \neq V_A \).

Theorem (A-Eklund-Kahle-Peterson)

A matrix \( A \) is diagonalizable if and only if \( Z_A = V_A \).
Eigenvectors of tensors

Set-up

- \( \{e_1, e_2, \ldots, e_n\} \): basis for \( \mathbb{C}^n \).
- For each tensor

\[
A := \sum_{i_1, i_2, \ldots, i_d=1}^n a_{i_1 i_2 \cdots i_d} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d} \in (\mathbb{C}^n) \otimes^d,
\]

define an element \( Ax^{d-1} \) of \( \mathbb{C}^n \otimes S^{d-1}(\mathbb{C}^n)^* \) by

\[
A x^{d-1} = \sum_{k=1}^n \left[ \sum_{i_1, \ldots, \hat{i}_j, \ldots, i_d=1}^n a_{i_0 \cdots i_{j-1} k i_{j+1} \cdots i_d} x_{i_1} \cdots \hat{x}_{i_j} \cdots x_{i_d} \right] e_k.
\]

Remark

If \( d = 2 \), then \( A x^1 = Ax \).
Eigenvectors of tensors (cont’d)

Definition

(1) $[v] \in \mathbb{P}(\mathbb{C}^n)$ is called an eigenpoint of $A$ if

$$A v^{d-1} \wedge v = 0.$$ 

(2) We call the closed subscheme $Z_A$ defined by $A x^{d-1} \wedge x$ the eigenscheme of $A$.

Theorem (Sturmfels-Cartwright, Oeding-Ottaviani)

If $A \in (\mathbb{C}^n)^{\otimes d}$ is generic, then $Z_A$ consists of

$$\delta := \frac{(d - 1)^n - 1}{d - 2} = \sum_{i=0}^{n-1} (d - 1)^i$$

distinct points.
Eigendiscriminant

Remarks

(1) The tensors of $(\mathbb{C}^n)^\otimes d$ whose eigenscheme is non-singular of dimension 0 form an open subset of $\mathbb{P}(\mathbb{C}^n)^\otimes d$.

(2) The complement $\Delta_{n,d}$ of the open subset in $\mathbb{P}(\mathbb{C}^n)^\otimes d$ is called the eigendiscriminant.

\[
\Delta_{n,d} := \left\{ [A] \in \mathbb{P}(\mathbb{C}^n)^\otimes d \mid |V_A| < \sum_{i=0}^{n-1} (d - 1)^i \right\}.
\]

Theorem (A-Seigal-Sturmfels, A-Lazarsfeld-Smith)

$\Delta_{n,d}$ is an irreducible hypersurface of $\mathbb{P}(\mathbb{C}^n)^\otimes d$ whose degree is

\[n(n-1)(d-1)^{n-1}.\]
Classification of eigenschemes in terms of partitions

(1) If $Z_A$ has dimension 0, then $\delta := \sum_{i=0}^{n-1} (d - 1)^i$ is its length.
(2) $Z_A = Z_1 \cup Z_2 \cup \cdots \cup Z_k$ : irreducible decomposition of $Z_A$.
(3) $\lambda_i$ : length of $Z_i$ for each $i \in \{1, 2, \ldots, k\}$.

**Definition**

We call the partition $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of $\delta$ the *partition* of $Z_A$.

**Example**

Consider the following tensor

\[
A := e_2 \otimes e_1^{d-1} + e_3 \otimes e_2^{d-1} + \cdots + e_n \otimes e_{n-1}^{d-1} \in (\mathbb{C})^d .
\]

If $I := \langle x_1, x_2, \ldots, x_{n-1} \rangle$ and if $P$ denotes the point defined by $I$, then $Z_A$ is a zero-dimensional closed subscheme of length $\delta$ supported at $P$. In particular, $(\delta)$ is the partition of $Z_A$. 
### Classification of eigenschemes in terms of partitions

#### Remarks

1. The previous example leads us to the question as to whether, for each of the remaining partitions $\lambda$ of $\delta$, there exists a tensor $A \in (\mathbb{C}^n)^{\otimes d}$ whose eigenscheme $Z_A$ has partition $\lambda$.

2. If $n = 1$ or if $d = 2$, then, for any partition $\lambda$ of $\delta$, there exists a tensor $A \in (\mathbb{C}^n)^{\otimes d}$ whose eigenscheme has partition $\lambda$.

#### Theorem (A-Galuppi-Gesmundo-Santarsiero-Seynnaeve)

If $n = 2$ and $d = 3$, then, for every partition $\lambda$ of $\delta = 7$, there exists a tensor $A \in (\mathbb{C}^n)^{\otimes d}$ whose eigenscheme has partition $\lambda$. 