Wheeler-DeWitt Equation in Spacekime

Yueyang Shen

Statistics Online Computational Resource
Computational Medicine & Bioinformatics
Biomedical Informatics and Data Science Training Program
University of Michigan

https://SOCR.umich.edu

petersyy@umich.edu

Joint work with Ivo D. Dinov (UM) & Milen V. Velev (BTU)

SCHOOL OF NURSING
STATISTICS ONLINE COMPUTATIONAL RESOURCE (SOCR)
UNIVERSITY OF MICHIGAN

Slides Online:
"SOCR News"

Outline

- ❑ Classical Wheeler-DeWitt Equation (WDE)
- ❑ Motivation
- ❑ Toy Models

- ❑ Complex-Time (*kime*)
- ❑ Kaluza Klein Approach
- ❑ Open Spacekime Problems

- ❑ Applications
  - ❑ Interdisciplinary Discussions

Classical Wheeler-DeWitt Equation

"Yet, the history of science has also convinced us to not dismiss ideas merely because they run counter to expectation. If we had, our most successful theory, quantum mechanics, which describes a reality governed by wholly peculiar waves of probability, would be buried in the trash bin of physics."

---Brian Greene

Historical Perspective

- Einstein's General Relativity (GR) 1915:

$$S = \int \left[\frac{1}{2\kappa}(R - 2\Lambda) + \mathcal{L}_{\mathrm{M}}\right]\sqrt{-g}\,\mathrm{d}^4x \quad \xrightarrow{\delta S = 0} \quad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}.$$

  - Spacetime grips mass, telling it how to move. Mass grips spacetime, telling it how to curve --- John Wheeler
- Planck postulate the quantum (1900):
  - Energy of electric radiation is quantized $E = nh\nu, n = 0,1,2, \ldots\ldots$
- Heisenberg and Schrodinger developed the foundations of quantum mechanics
  - Probabilistic time evolution (1925) $\hat{H}\psi = ih\frac{\partial}{\partial t}\psi$
  - Heisenberg Uncertainty Principle (1927) $\Delta_x \Delta_p \geq \frac{h}{4\pi}$
- ADM (Arnowitt, Deser, Misner) formalism (Hamiltonian Gravity, 1959)[1]:

$$H = -\sqrt{g}\left[{}^{(3)}R + g^{-1}\left(\frac{1}{2}\pi^2 - \pi^{ij}\pi_{ij}\right)\right]$$

  π_{ij} is the conjugate momentum (density)
  - Spacetime foliation Σ_t
- Wheeler-Dewitt Equation (1967) [2]:
  - Promoting to quantum Hamiltonian $\hat{\mathcal{H}}$ (Timelessness) $\hat{\mathcal{H}}\psi = 0$

[1] Arnowitt, Deser, Misner, *Dynamical structure and definition of energy in general relativity.* Physical Review, 1959. **116**(5): p. 1322.
[2] DeWitt, B.S., *Quantum theory of gravity. I. The canonical theory.* Physical Review, 1967. **160**(5): p. 1113.

Motivation: The foundation of quantum cosmology

Q: WDE significance

- The Wheeler-DeWitt geometrodynamic equation is perhaps the oldest approach to quantize gravity (Pre superstring revolution)
  - Unification is the key enterprise of modern physical advances[3]
  - First Superstring revolution(1984) (John Schwarz Michael Green)[4]

Q: The meaning of WDE

- Canonically speaking, everything has already happened in the universe[5]
- Canonical WDE applies quantum world view to the universe as a whole, but the unification introduces many problems including: regularization, problem of time, and mathematical difficulties of functional differential equation[5].

Q: WDE applications to study quantum cosmology

- Quantum Cosmology: Bryce DeWitt, Charles Misner, John Wheeler 1960s
  - Took off by Hartle and Hawking: Wavefunctions of Universe(1980)[6]
- Quantum Cosmology studies the problem of the birth of our universe with various initial conditions by apply quantum theory's principles to classical description of the universe given by relativistic cosmology.

[3] https://www.smithsonianmag.com/science-nature/string-theory-about-unravel-180953637/
[4] Michael B. Green, John H. Schwarz, Anomaly cancellations in supersymmetric D = 10 gauge theory and superstring theory, Physics Letters B, Volume 149, Issues 1–3, 1984, Pages 117-122, ISSN 03702693,https://doi.org/10.1016/0370-2693(84)91565-X.
[5] Anderson, E., *Problem of time in quantum gravity.* Annalen der Physik, 2012. **524**(12): p. 757-786.
[6] Hartle, J.B. and S.W. Hawking, *Wave function of the Universe.* Physical Review D, 1983. **28**(12): p. 2960-2975.

WDE – Canonical Derivation

From a canonical point of view, the derivation of the classical Wheeler Dewitt equation (WDE) from quantization of classical Einstein Hilbert action is a threefold procedure:

- ❑ Start out with the general covariant theory, express the canonical momenta.
- ❑ Write out Hamiltonian using Legendre transformation
- ❑ Promote induced metric and momenta to operators and Poisson bracket to commutators [7].

The Hamiltonian is achieved by ADM formalism which foliates spacetime into a family of time indexed spatial slices families resulting in a time independent Hamiltonian constraint ("Wheeler-DeWitt equation") and diffeomorphism constraint [8, 9].

[7] Kiefer, C., *Quantum geometrodynamics: whence, whither?* General relativity and gravitation, 2009. 41(4): p. 877-901.
[8] Hersch, M., *The Wheeler-DeWitt equation.* 2017.
[9] Kiefer, C., *Quantum Gravity.* 2007: Oxford University Press

Toy Model of Quantum Cosmology

- We may play around with the Wheeler-DeWitt equation by gauge fixing extra degrees of freedom and obtaining a well-posed differential equation
- Minisuperspace model with flat Friedmann universe[9]:

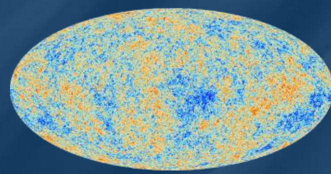


Figure 1. Cosmic microwave background[10]

$$ds^2 = -N(t)^2\,dt^2 + a(t)^2 d\Omega_3^2$$

$$d\Omega_3^2 = dy^2 + sin^2 y(d\theta^2 + sin^2\theta d\varphi^2)$$

$$S = \int_M (d^4x(R - 2\Lambda)\sqrt{g}) + \int_{\partial M} (d^3x\sqrt{h}K)$$

$$R = \frac{6}{N^2}\left(-\frac{\dot{N}\dot{a}}{Na} + \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) + \frac{6}{a^2} \qquad K = \frac{3\dot{a}}{Na}$$

$$\sqrt{h}d^3x = a^3 sin^2 y\, sin\theta d\theta dy d\varphi$$

- Gravitational section

$$S = \frac{1}{2}\int dt\,\mathrm{N}\left(-\frac{a\dot{a}^2}{N^2} + a - \Lambda\frac{a^3}{3}\right)$$

- Matter section

$$S_m = \frac{1}{2}\int dt\,\mathrm{N}a^3\left(\frac{\dot{\phi}^2}{N^2} - m^2\phi^2\right)$$

- Derive the momenta and apply Legendre transformation

$$H = \frac{N}{2}\left(-\frac{2\pi G}{3}\frac{p_a^2}{2\pi^2 a} + \frac{p_\phi^2}{a^3} - \frac{3}{8\pi G}2\pi^2 a + \frac{2\pi^2\Lambda}{8\pi G}a^3 + m^2\phi^2 a^3\right)$$

- H=0 Friedman equation (N=1)

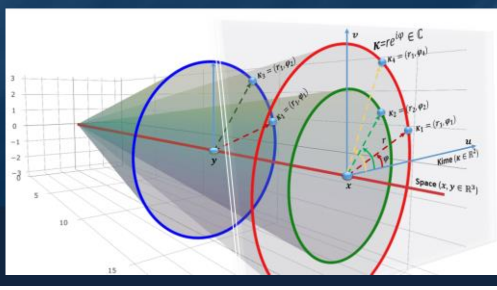
$$\dot{a}^2 = -1 + a^2\left(\dot{\phi}^2 + \frac{\Lambda}{3} + m^2\phi^2\right)$$

[9] Kiefer, C., *Quantum Gravity*. 2007: Oxford University Press
[10] https://en.wikipedia.org/wiki/Friedmann-Lemaître-Robertson-Walker_metric

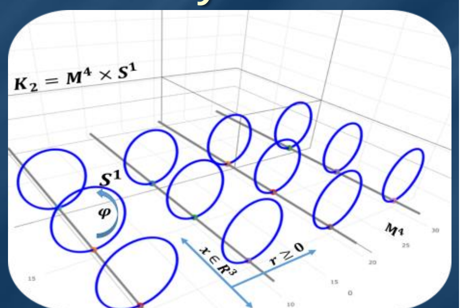
Complex-Time (*kime*) & Spacekime Foundations

Complex-Time (*Kime*)

- At a given spatial location, $\boldsymbol{x}$, complex time (*kime*) is defined by $\kappa = re^{i\varphi} \in \mathbb{C}$, where:
  - the magnitude represents the longitudinal events order ($r > 0$) and characterizes the longitudinal displacement in time, and
  - event phase ($-\pi \le \varphi < \pi$) is an angular displacement, or event direction
- There are multiple alternative parametrizations of kime in the complex plane
- Space-kime manifold is $\mathbb{R}^3 \times \mathbb{C}$:
  - $(\boldsymbol{x}, k_1)$ and $(\boldsymbol{x}, k_4)$ have the same spacetime representation, but different spacekime coordinates,
  - $(\boldsymbol{x}, k_1)$ and $(\boldsymbol{y}, k_1)$ share the same kime, but represent different spatial locations,
  - $(\boldsymbol{x}, k_2)$ and $(\boldsymbol{x}, k_3)$ have the same spatial-locations and kime-directions, but appear sequentially in order, $r_2 < r_1$.



Kaluza-Klein Theory



- Theodor Kaluza (1921) developed a math extension of the classical general relativity theory to 5D. This included the metric, the field equations, the equations of motion, the stress-energy tensor, and the cylinder condition. Physicist Oskar Klein (1926) interpreted Kaluza's 3D+2D theory in quantum mechanical space and proposed that the fifth dimension was curled up and microscopic.
- The topology of the 5D Kaluza-Klein spacetime is $K_2 \cong M^4 \times S^1$, where M^4 is a 4D Minkowski spacetime and S^1 is a circle (non-traversable).
- Kaluza Klein approach (1921) (Unify GR and electromagnetism):
  - High dimensional idea and compactify on a circle and do physics in the lower dimensional space
  - 5 dimensional general covariance -> 4 dimensional gauge invariance of the vector field

Derivation of WDE in Spacekime (4+1 decomposition)

- Start with general covariant theory

$$S = \int_M \left(d^{n+1}x \frac{1}{2\kappa}(R - 2\Lambda)\sqrt{g}\right) + \frac{\varepsilon}{\kappa}\int_{\partial M}\left(d^n x\sqrt{h}K\right)$$

- Gauss-Codazzi Equation

$$R(g) = R(h) + \epsilon\left(K^2 - K^{\alpha\beta}K_{\alpha\beta}\right) + 2\epsilon\nabla_\alpha\Big(\underbrace{n^\beta\nabla_\beta n^\alpha}_{a^\alpha} - n^\alpha \underbrace{\nabla_\beta n^\beta}_{K}\Big) = R(h) + \epsilon\left(K^2 - K^{\alpha\beta}K_{\alpha\beta}\right) + 2\epsilon\nabla_\alpha a^\alpha - 2\epsilon\nabla_\alpha(n^\alpha K)$$

- Bulk Lagrangian

$$\mathcal{L}_M = \frac{1}{2\kappa}\left(R(h) - 2\Lambda + \epsilon\left(K^2 - K^{\alpha\beta}K_{\alpha\beta}\right)\right)$$

$$\Pi^{\alpha\beta} = \frac{\partial\mathcal{L}_M}{\partial(\mathcal{L}_n h_{\alpha\beta})} = \frac{\epsilon}{2\kappa}\left(h^{\alpha\beta}K - K^{\alpha\beta}\right)\ ;\ \Pi_{\alpha\beta} = \frac{\epsilon}{2\kappa}\left(h_{\alpha\beta}K - K_{\alpha\beta}\right),$$

$$\Pi = h_{\alpha\beta}\Pi^{\alpha\beta} = \Pi^\alpha_\alpha = \frac{\epsilon(n-1)}{2\kappa}K,$$

$$H(p,q,\kappa_1,\kappa_2) = H\underbrace{(\Pi_{\alpha\beta})}_{\text{phase space}} = -\epsilon\kappa^2\left(\Pi^{\alpha\beta}\Pi_{\alpha\beta} - \frac{1}{n-1}\Pi^2\right) + R - 2\Lambda.$$

$$\widehat{\mathcal{H}} = -\epsilon\kappa^2 \!:\!\left(\widehat{\Pi}^{\alpha\beta}\widehat{\Pi}_{\alpha\beta} - \frac{1}{n-1}\widehat{\Pi}^2\right)\!: + R - 2\Lambda$$

Interdisciplinary Discussions

If we don't do gravity and simply consider five-dimensional scalar field,

$$\mathcal{L} = \frac{1}{2}\partial_M\varphi\partial^M\varphi - \frac{m_5^2}{2}\varphi^2$$

Expanding on the Fourier modes,

$$\underbrace{\varphi(X^\mu, X^5 \equiv y)}_{\text{five dimensional klein gordon}} = \sum_n \varphi_n(x^\mu)e^{\frac{iny}{R}}$$

Where the 5th dimension satisfy $X^5 = X^5 + 2\pi R$. Upon varying the action, the equation of motion satisfy

$$\left(\partial_M\partial^M - m_5^2\right)\varphi(X^\mu, X^5 \equiv y) = 0$$

$$\Rightarrow \left(\partial_\mu\partial^\mu + (p^5)^2 - m_5^2\right)\varphi_n(X^\mu) = 0$$

$$p^5 = \frac{1}{i}\partial_y = \frac{n}{R}$$

$$\left(\partial_\mu\partial^\mu - m_{4,n}^2\right)\varphi_n(X^\mu) = 0, \qquad m_{4,n}^2 = m_5^2 + \left(\frac{n}{R}\right)^2$$

Zero excitation mode: $\varphi_0(X^\mu)$: $m_{4,0}^2 = m_5^2$

Higher excitation mode: $\underbrace{\varphi_{n>0}(X^\mu)}_{\text{Kaluza Klein tower}}$: $m_{4,0}^2 = m_5^2 + \left(\frac{n}{R}\right)^2$

Ultra-hyperbolic equation in disguise[11]

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} - \left(\frac{\partial^2 u}{\partial t_1^2} + \frac{\partial^2 u}{\partial t_1^2} + \cdots + \frac{\partial^2 u}{\partial t_n^2}\right) = 0.$$

- Ultrahyperbolic equations represent partial differential equations of an unknown scalar-valued wavefunction, $u = u(x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n)$, of a set of n spatial variables, $\{x_1, x_2, \cdots, x_n\}$, and another set of n temporal variables $\{t_1, t_2, \cdots, t_n\}$
- Ultrahyperbolic equations naturally extend spacetime waves propagating through a medium in 4D to 5D processes, and higher dimensional spaces, where two or more time-like dynamic variable govern the state of the system [12]

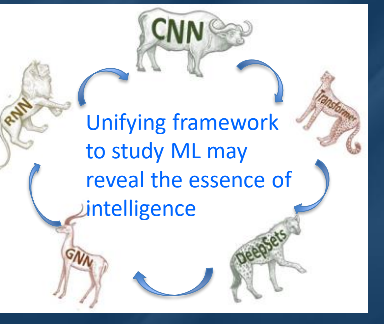
[11] Wang, Y., et al., *Determinism, Well-posedness, and Applications of the Ultrahyperbolic Wave Equation in Spacekime.* in review, 2021.
[12] Bars, I. and C. Kounnas, *String and particle with two times.* Physical Review D, 1997. 56(6): p. 3664

Unification in the era of Data Science and AI

- Einstein: General Relativity---the evolution of geometry

  Physical unification (forces) probes into the fundamentals of our perceived reality

- Quantum: The study of the micros
- Geometric Deep Learning aims unify current ML algorithms from the perspectives of symmetry and invariance [13-14].



AI zoo (Bronstein)[13]

- An invariance principle articulates symmetries of physical systems, and is connected to laws of physics. In spacetime, Noether proposed that the conservation of angular momentum is related to rotational symmetries (rotational invariance); while the conservation of energy is related to translational symmetries in time (time invariance); and the conservation of linear momentum is related to translational symmetries (translational invariance).

- An invariant function remains untouched under the action of the group [13] (CNN is translational invariant), future discoveries of the invariances and symmetries are essential for unifying principles in deep network machine learning and artificial intelligence.

[13] https://towardsdatascience.com/geometric-foundations-of-deep-learning-94cdd45b451d
[14] Bars, I. and C. Kounnas, *String and particle with two times.* Physical Review D, 1997. 56(6): p. 3664

Current Open problems

- There is currently no known mathematically rigorous framework to quantize a theory ("canonical")
  - Many classical results are broken by quantum effects (Weyl anomaly).
  - Fadeev Popov ghost
  - Some older approaches (buzz words): lightcone quantization, old covariant quantization, BRST quantization…

Future/Ongoing Research directions

- 3+2 formalism
  - Vielbein/Tetrad formalism: Dual Null Coordinates[13-15]
  - Constructing geometrically succinct objects
- Factorizable spacetime m+n formalism[16]

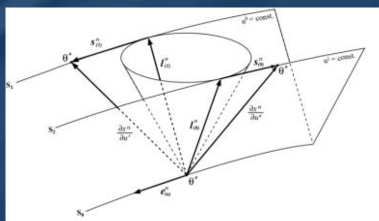


Figure Double null foliation[13]

[13] Brady, et al., *Covariant double-null dynamics: 2+ 2-splitting of the Einstein equations.* Classical and Quantum Gravity, 1996. **13**(8): p. 2211.
[14] Vickers, *Double null Hamiltonian dynamics and the gravitational degrees of freedom.* General Relativity and Gravitation, 2011. **43**(12): p. 3411
[15] Lehner, et al., *Gravitational action with null boundaries.* Physical review. D, 2016. **94**(8).
[16] Ripley, Yagi, *Black hole perturbation under a 2+ 2 decomposition in the action.* Physical Review D, 2018. **97**(2): p. 024009.

Acknowledgments

Slides Online: "SOCR News"

Funding

NIH UL1TR002240, R01CA233487, R01MH121079, R01MH126137, T32GM141746 (BIDS-TP)
NSF: 1916425, 1734853, 1636840, 1416953, 0716055, 1023115

Collaborators

- **SOCR: Milen Velev, Yueyang Shen, Daxuan Deng, Zijing Li, Yongkai Qiu, Zhe Yin, Yufei Yang, Yuxin Wang, Rongqian Zhang, Yuyao Liu, Yupeng Zhang, Yunjie Guo**
- **UMich MIDAS/MNORC/AD/PD Centers**: Chuck Burant, Kayvan Najarian, Stephen Goutman, Stephen Strobbe, Hiroko Dodge, Chris Monk, Issam El Naqa, HV Jagadish, Brian Athey

https://SOCR.umich.edu

Thanks!