Node Centralities in Multilayer Networks

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Key References

- Background on multilayer networks
  - Review article: Mikko Kivelä, Alex Arenas, Marc Barthelemy, James P. Gleeson, Yamir Moreno, & MAP [2014], "Multilayer Networks", *Journal of Complex Networks*, Vol. 2, No. 3: 203–271
  - Manlio De Domenico, Albert Solé-Ribalta, Emanuele Cozzo, M. Kivelä, Y. Moreno, MAP, Sergio Gómez, and A. Arenas [2013], "Mathematical Formulation of Multilayer Networks", *Physical Review X*, Vol. 3, No. 4: 041022
- Multilayer eigenvector-based centralities
  - Dane Taylor, Sean A. Myers, Aaron Clauset, MAP, & P. J. Mucha, "Eigenvector-Based Centrality Measures for Temporal Networks", *Multiscale Modeling and Simulation: A SIAM Interdisciplinary Journal*, Vol. 15, No. 15, No. 1: 537–574
  - D. Taylor, MAP, & P. J. Mucha [2019], "Supracentrality Analysis of Temporal Networks with Directed Interlayer Coupling", in *Temporal Network Theory* (Petter Holme & Jari Saramäki, Eds.), pages 325–344; Springer International Publishing
  - D. Taylor, MAP, & P. J. Mucha [2021], "Tunable Eigenvector-Based Centralities for Multiplex and Temporal Networks", *Multiscale Modeling and Simulation: A SIAM Interdisciplinary Journal*, Vol. 19, No. 1: 113–147

Outline

- “Centralities” of nodes in networks
- Multilayer networks
- Eigenvector-based centralities in multilayer networks
- Conclusions

"Centralities" of Nodes in Networks

Calculating centralities gives a way to measure the importance of nodes, edges, and other structures in networks.

“Centrality” Measures

- Calculating centralities gives a way to measure the importance of nodes, edges, and other structures in networks.
  - This talk: node centralities
- Degree = number of adjacent nodes (e.g., number of friends)
- Betweenness centralities = on many short paths
- Eigenvector centrality = a node is important if it is adjacent to important nodes

Eigenvector-Based Centralities

- Examples
  - Eigenvector centrality
    - Leading-eigenvector solution of the eigenvalue problem $Av = \lambda v$
  - Hubs and authorities
    - Leading-eigenvector solution: $x = aAy$; $y = bA^T x$ ➔ $A^T A y = \lambda y$ & $AA^T x = \lambda x$, where $\lambda = 1/(ab)$
    - Node i has hub centrality x_i and authority centrality y_i
  - PageRank
    - Review article: David F. Gleich [2015], "PageRank Beyond the Web", *SIAM Review*, Vol. 57, No. 3: 321-363

Let $P_{i,j}$ be the probability of transitioning from page j to page i (or, more generally, from "thing j" to "thing i"). The stationary distribution of the PageRank Markov chain is called the PageRank vector $\mathbf{x}$, which is the solution of the eigenvalue problem

$$(\alpha \mathbf{P} + (1-\alpha)\mathbf{v}\mathbf{e}^T)\mathbf{x} = \mathbf{x}. \quad (2.1)$$

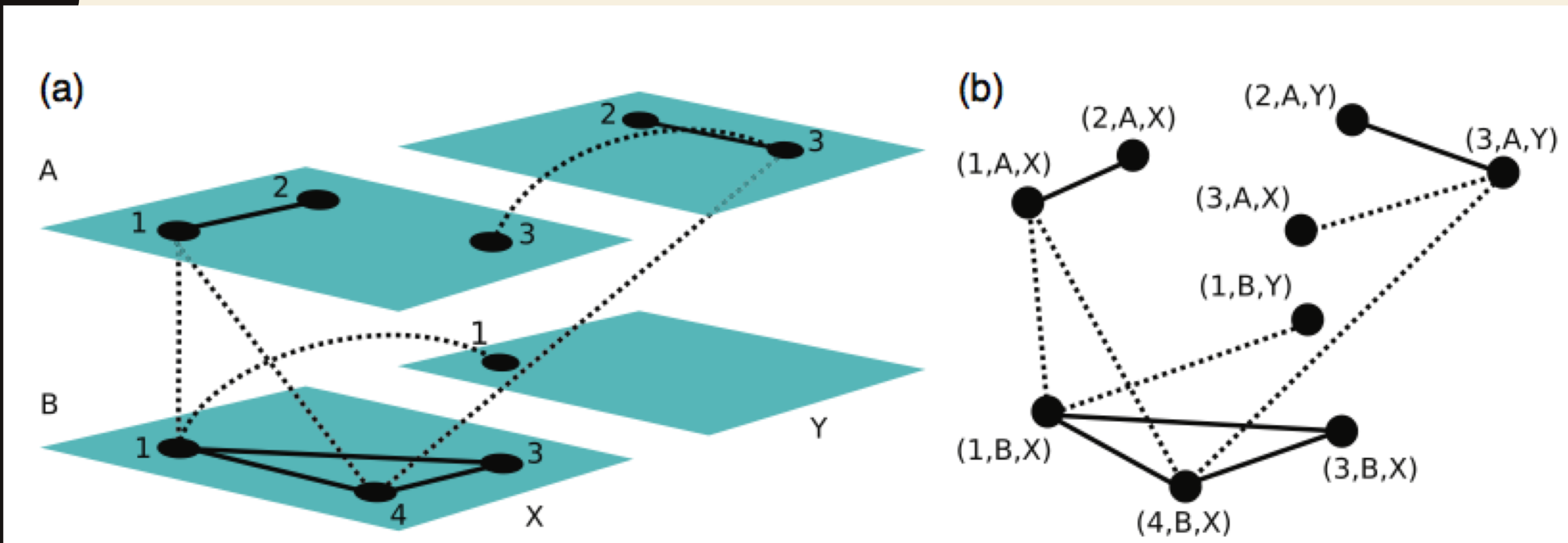
Many take this eigensystem as the definition of PageRank (Langville and Meyer, 2006). We prefer the following definition instead.

- Note: Other centralities (e.g., Katz centrality) arise from eigenvectors, but the centrality vector itself is not a solution of an eigenvalue problem.

Multilayer Networks

A general type of network for studying multirelational networks, temporal networks, and other types of networks

Multilayer Network



Example: Node-Colored Network

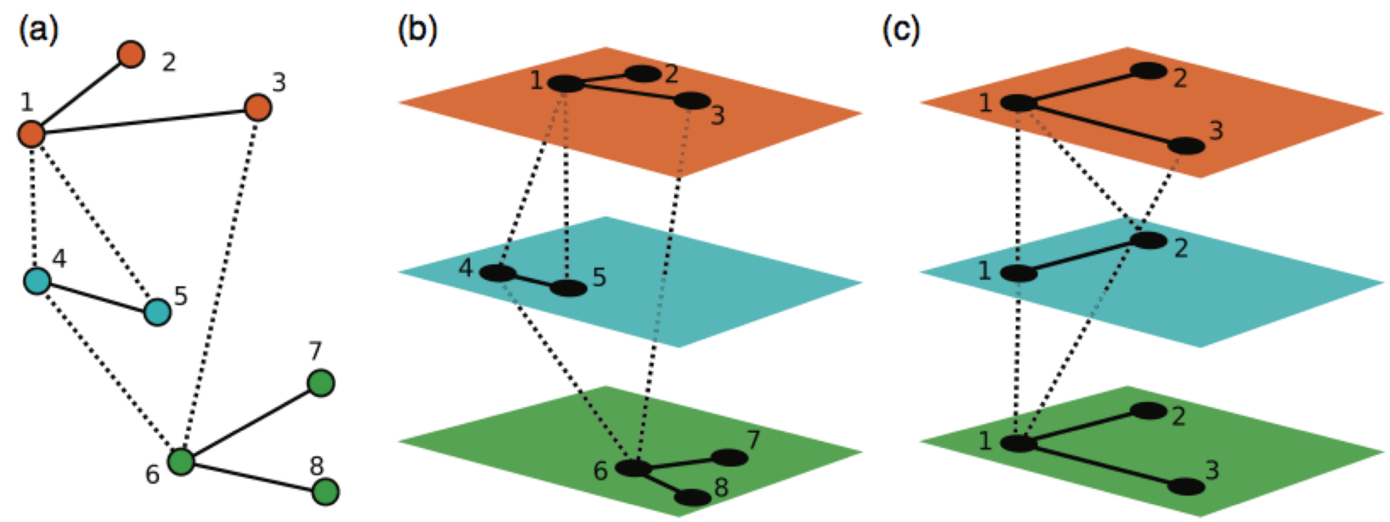
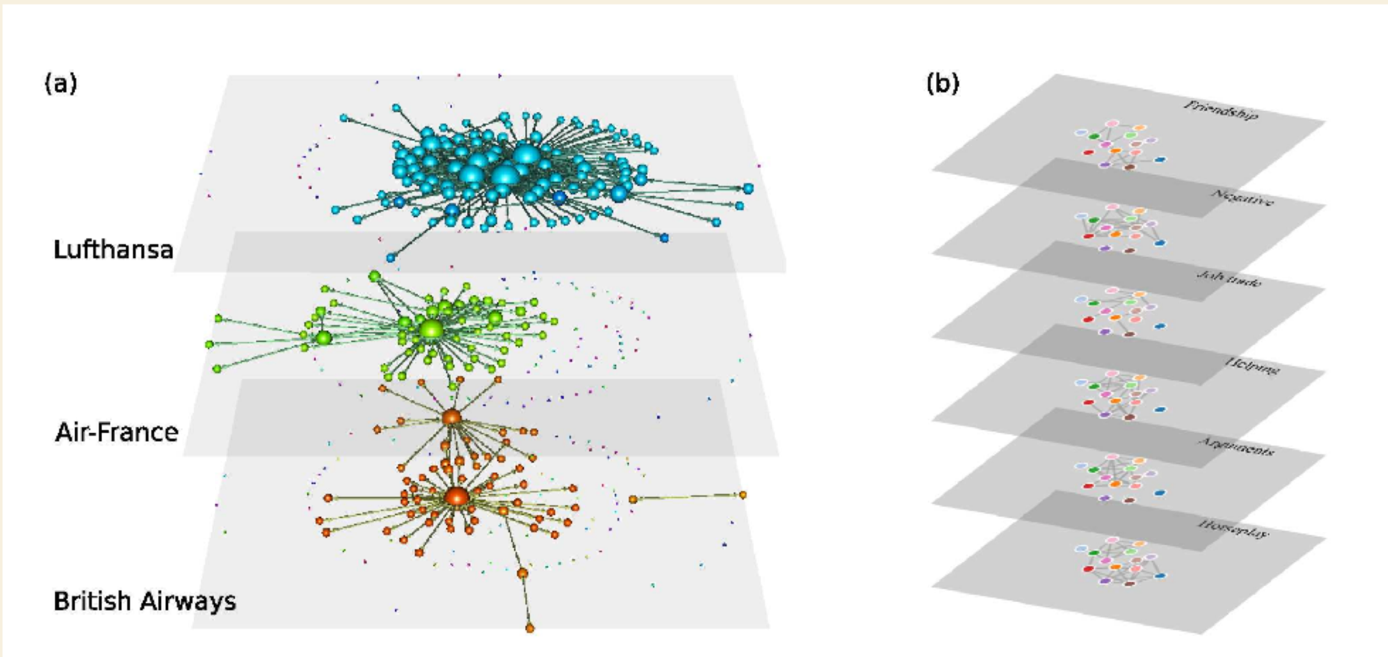


FIG. 5. (a) An example of a node-coloured network (i.e. an interconnected network, a network of networks etc.). (b) Representation of the same node-coloured network using our multilayer network formalism. We keep the node names from the original network. (c) Alternative representation of the same node-coloured network in our multilayer network formalism. This time, we use consecutive integers starting from 1 to name the nodes in each layer, so we also need to include the identity of the layer to uniquely specify each node.

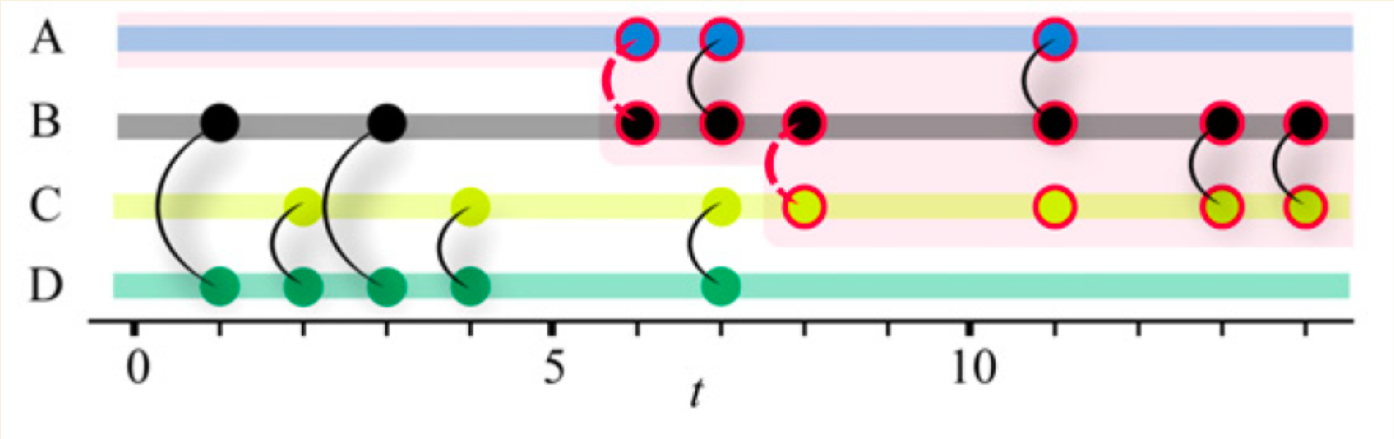
Example: Multiplex Network

- An old idea from the social networks literature
  - Multirelational network (edge coloring)



Example: Temporal Network

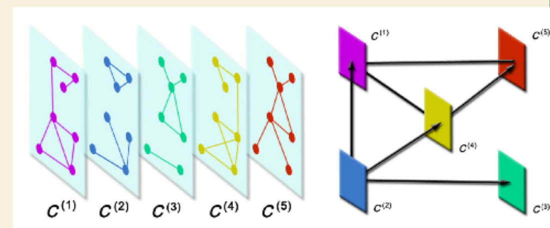
- Time-dependent networks
- In a multilayer representation of a temporal network, each layer may be a point in time, an aggregation over some time period, etc.



Schematic from: P. Holme & J. Saramäki [2012], "Temporal Networks", *Physics Reports*, Vol. 519: 97-125

General Form of a Multilayer Network

- Definition of a multilayer network M
  - $M = (V_M, E_M, V, L)$
    - V: set of nodes ("entities")
      - As in ordinary graphs
    - L: sequence of sets of possible layers
      - One set for each additional "aspect" $d \geq 0$ beyond an ordinary network (example: $d = 1$ in schematic on this page)
    - V_M: set of tuples that represent node-layers
    - E_M: multilayer edge set that connects these tuples



- Note: One can weight edges in the usual way

Tensorial Representation

- Adjacency tensor for unweighted case:

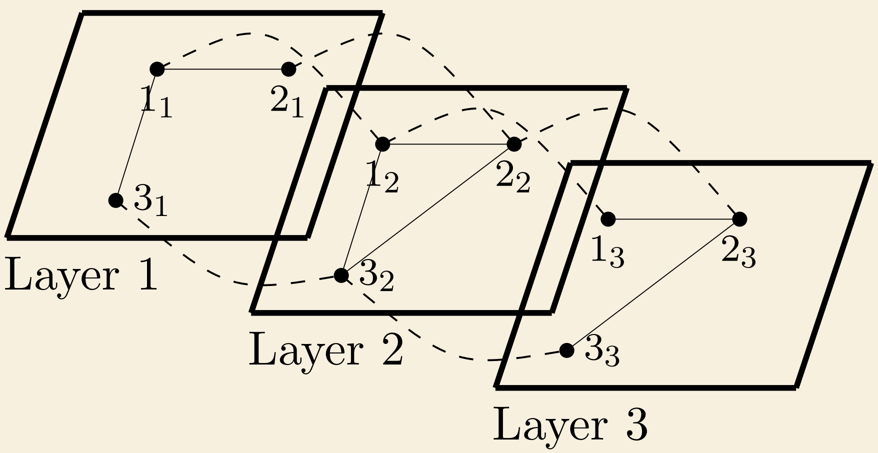
$$\mathcal{A} \in \{0,1\}^{|V| \times |V| \times |\mathbf{L}_1| \times |\mathbf{L}_1| \times \cdots \times |\mathbf{L}_d| \times |\mathbf{L}_d|}$$

- Elements of adjacency tensor:
  - $A_{uv\boldsymbol{\alpha\beta}} = A_{uv\alpha 1\beta 1 \ldots \alpha d\beta d} = 1$ if and only if $((u,\boldsymbol{\alpha}), (v,\boldsymbol{\beta}))$ is an element of E_M (otherwise, $A_{uv\boldsymbol{\alpha\beta}} = 0$)

- Important note: 'padding' layers with empty nodes
  - One distinguishes between a node not present in a layer and nodes existing but edges not present. This is important for normalization for many quantities.
    - "Missing edges" versus "forbidden edges"

“Flattened” Multilayer Networks

(supra-adjacency representation)



$$\longleftrightarrow \left[\begin{array}{ccc|ccc|ccc}
0 & 1 & 1 & \omega & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 \\
\hline
\omega & 0 & 0 & 0 & 1 & 1 & \omega & 0 & 0 \\
0 & \omega & 0 & 1 & 0 & 1 & 0 & \omega & 0 \\
0 & 0 & \omega & 1 & 1 & 0 & 0 & 0 & \omega \\
\hline
0 & 0 & 0 & \omega & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \omega & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & \omega & 0 & 1 & 0
\end{array}\right]$$

- Schematic from M. Bazzi, MAP, S. Williams, M. McDonald, D. J. Fenn, & S. D. Howison [2016] *Multiscale Modeling and Simulation: A SIAM Interdisciplinary Journal*, **14**(1): 1-41

Eigenvector-Based Centralities in Multilayer Networks

We generalize eigenvector-based centralities from ordinary graphs to multilayer networks.

Overview of the Three Central Publications

- 2017 paper: Introduces a principled generalization of eigenvector-based centralities for temporal networks (using multilayer representations of temporal networks)
  - Assumes symmetric, adjacent-in-time coupling of temporal layers
  - Perturbation theory for strong interlayer coupling

- 2019 book chapter: Generalizes 2017 paper to consider directed interlayer edges
  - Multilayer PageRank has both ordinary teleportation and “layer teleportation”

- 2021 paper: Generalizes the previous work to very general forms of interlayer coupling, allowing both temporal eigenvector-based centralities and multiplex eigenvector-based centralities
  - Perturbation theory for both strong interlayer coupling and weak interlayer coupling

2017 Paper: Construct a Supracentrality Matrix

$$\mathbb{M}(\epsilon) = \begin{bmatrix} \epsilon\mathbf{M}^{(1)} & \mathbf{I} & 0 & \cdots \\ \mathbf{I} & \epsilon\mathbf{M}^{(2)} & \mathbf{I} & \ddots \\ 0 & \mathbf{I} & \epsilon\mathbf{M}^{(3)} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

- E.g. $\mathbf{M}^{(t)} = \mathbf{A}^{(t)}[\mathbf{A}^{(t)}]^T$ to examine hubs and authorities (t indexes the layers)
- $\varepsilon = 1/\omega$
- A singular perturbation from the $\varepsilon \rightarrow 0$ (strong coupling) limit yields a time-independent centrality and then a “first mover” perturbation term (and higher-order corrections)

Singular Perturbation Expansion

$$\boxed{\begin{aligned} \mathbb{M}(\epsilon) &= \mathbb{B} + \epsilon\mathbb{G} \\ \mathbb{B} &= \boldsymbol{A}^{(\mathrm{chain})} \otimes \mathbf{I} \\ \mathbb{G} &= \mathrm{diag}[\mathbf{M}^{(1)}, \ldots, \mathbf{M}^{(T)}] \end{aligned}}$$

where $\otimes$ denotes the Kronecker product and $\boldsymbol{A}^{(\mathrm{chain})}$ is the $T \times T$ adjacency matrix of an undirected "bucket brigade" (or "chain") network whose T nodes are each adjacent to their nearest neighbors along an undirected chain. In this bucket brigade,

$$\boxed{\begin{aligned} &\lambda_{\max}(\epsilon)\mathbb{v}(\epsilon) = \mathbb{M}(\epsilon)\mathbb{v}(\epsilon) = \mathbb{B}\mathbb{v}(\epsilon) + \epsilon\mathbb{G}\mathbb{v}(\epsilon) \\ &\lambda_{\max}(\epsilon) = \lambda_0 + \epsilon\lambda_1 + \cdots \quad \mathbb{v}(\epsilon) = \mathbb{v}_0 + \epsilon\mathbb{v}_1 + \cdots \end{aligned}}$$

$$\lambda^{(\mathrm{chain})} = 2\cos\left(\frac{n\pi}{T+1}\right),$$

$$\boldsymbol{u}^{(\mathrm{chain})} = \frac{1}{\sqrt{\gamma_n}}\left[\sin\left(\frac{n\pi}{T+1}\right), \sin\left(\frac{2n\pi}{T+1}\right), \ldots, \sin\left(\frac{Tn\pi}{T+1}\right)\right]^T,$$

0^{th} Order Expansion and Time-Averaged Centrality

$$\mathbf{X}^{(1)}\boldsymbol{\alpha} = \lambda_1 \boldsymbol{\alpha}$$

α_i is the time-averaged centrality for entity i

$$X_{ij}^{(1)} = \mathbb{u}_i^T \mathbb{P}^T \mathbb{G} \mathbb{P} \mathbb{u}_j = \gamma_1^{-1} \sum_t M_{ij}^{(t)} \sin^2\left(\frac{\pi t}{T+1}\right)$$

One derives expressions for higher-order corrections in a similar way. The coefficient of the first correction gives a first-mover score.

Mathematics PhD Exchange Network

- Mathematics department rankings change with time, so we want to use centrality measures that change with time
- Multilayer network with adjacency tensor elements A_{ijst}
  - Directed intralayer edge from university i to university j at time t for a specific person's PhD granted at time t for a person who later advised a student at i (multi-edges give weights)
    - E.g. Peter Mucha yields a Princeton➔Georgia Tech edge and a Princeton➔UNC Chapel Hill edge for t = 1998
      - Note: No Princeton➔Dartmouth edge (time delay)
  - Use a multilayer network with "diagonal" and ordinal interlayer coupling
  - 231 US universities, T = 65 time layers (1946-2010)

Mathematics Genealogy Project

Peter John Mucha

MathSciNet

Ph.D. Princeton University 1998

Dissertation: *On Zero Reynolds Number Microhydrodynamics of Particulate Suspensions*

Advisor 1: Isaac Goldhirsch
Advisor 2: Steven Alan Orszag

Students:
Click here to see the students ordered by family name.

Name	School	Year	Descendants
Carlson, Mark	Georgia Institute of Technology	2004	
Hohenegger, Christel	Georgia Institute of Technology	2006	1
Shi, Feng	The University of North Carolina at Chapel Hill	2013	
Wang, Simi	The University of North Carolina at Chapel Hill	2014	
Lee, Hsuan-Wei	The University of North Carolina at Chapel Hill	2016	
Heroy, Samuel	The University of North Carolina at Chapel Hill	2018	
Stanley, Natalie	The University of North Carolina at Chapel Hill	2018	
Weir, William	The University of North Carolina at Chapel Hill	2020	

Math Departments: Best Authorities

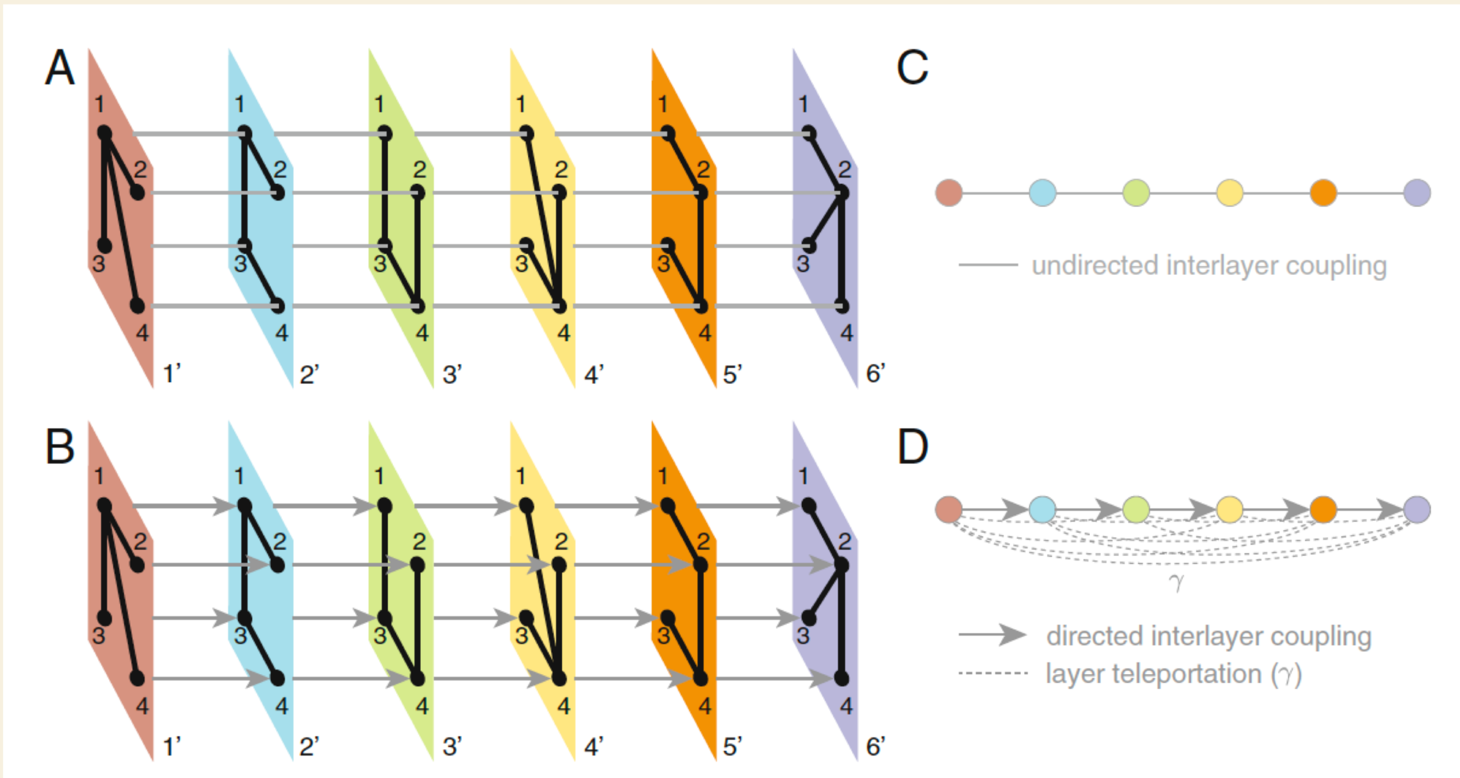
Top Time-Averaged Centralities

Rank	University	α_i
1	MIT	0.6685
2	Berkeley	0.2722
3	Stanford	0.2295
4	Princeton	0.1803
5	Illinois	0.1645
6	Cornell	0.1642
7	Harvard	0.1628
8	UW	0.1590
9	Michigan	0.1521
10	UCLA	0.1456

Top First-Order Mover Scores

Rank	University	m_i
1	MIT	688.62
2	Berkeley	299.07
3	Princeton	248.72
4	Stanford	241.71
5	Georgia Tech	189.34
6	Maryland	186.65
7	Harvard	185.34
8	CUNY	182.59
9	Cornell	180.50
10	Yale	159.11

2019 Book Chapter: Directed Interlayer Coupling
["Node Teleportation" and "Layer Teleportation"]



2021 Paper: Both Temporal and Multiplex Networks

- Much more general form of interlayer coupling
- We do perturbation theory for both weak and strong coupling

(3.1)

$$\mathbb{C}(\omega) = \hat{\mathbb{C}} + \omega\hat{\mathbb{A}} = \begin{bmatrix} \mathbf{C}^{(1)} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{C}^{(2)} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{C}^{(3)} & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} + \omega \begin{bmatrix} \tilde{A}_{11}\mathbf{I} & \tilde{A}_{12}\mathbf{I} & \tilde{A}_{13}\mathbf{I} & \dots \\ \tilde{A}_{21}\mathbf{I} & \tilde{A}_{22}\mathbf{I} & \tilde{A}_{23}\mathbf{I} & \dots \\ \tilde{A}_{31}\mathbf{I} & \tilde{A}_{32}\mathbf{I} & \tilde{A}_{33}\mathbf{I} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $\hat{\mathbb{C}} = \mathrm{diag}[\mathbf{C}^{(1)}, \dots, \mathbf{C}^{(T)}]$ *and* $\hat{\mathbb{A}} = \tilde{\boldsymbol{A}} \otimes \mathbf{I}$ *denotes the Kronecker product of* $\tilde{\boldsymbol{A}}$ *and* $\mathbf{I}$.

Marginal Node and Layer Centralities

DEFINITION 3.3 (joint centralities of node-layer pairs [110]). *Let* $\mathbb{C}(\omega)$ *be a supra-centrality matrix given by Definition* 3.1, *and let* $\mathbb{v}(\omega)$ *be its right dominant eigenvector. We encode the* joint centrality *of node* i *in layer* t *via the* $N \times T$ *matrix* $\mathbf{W}(\omega)$ *with entries*

$$W_{it}(\omega) = \mathbb{v}_{N(t-1)+i}(\omega)\,. \tag{3.3}$$

REMARK 3.4. *We refer to* $W_{it}(\omega)$ *as a "joint centrality" because it reflects the importance of both node* i *and layer* t.

DEFINITION 3.5 (marginal centralities of nodes and layers [110]). *Let* $\mathbf{W}(\omega)$ *encode the joint centralities of Definition* 3.3. *We define the* marginal layer centrality *(MLC)* $x_t(\omega)$ *and* marginal node centrality *(MNC)* $\hat{x}_i(\omega)$ *by*

$$x_t(\omega) = \sum_i W_{it}(\omega)\,, \quad \hat{x}_i(\omega) = \sum_t W_{it}(\omega)\,. \tag{3.4}$$

- We also examine conditional centralities (nodes conditioned on layers, and vice versa)

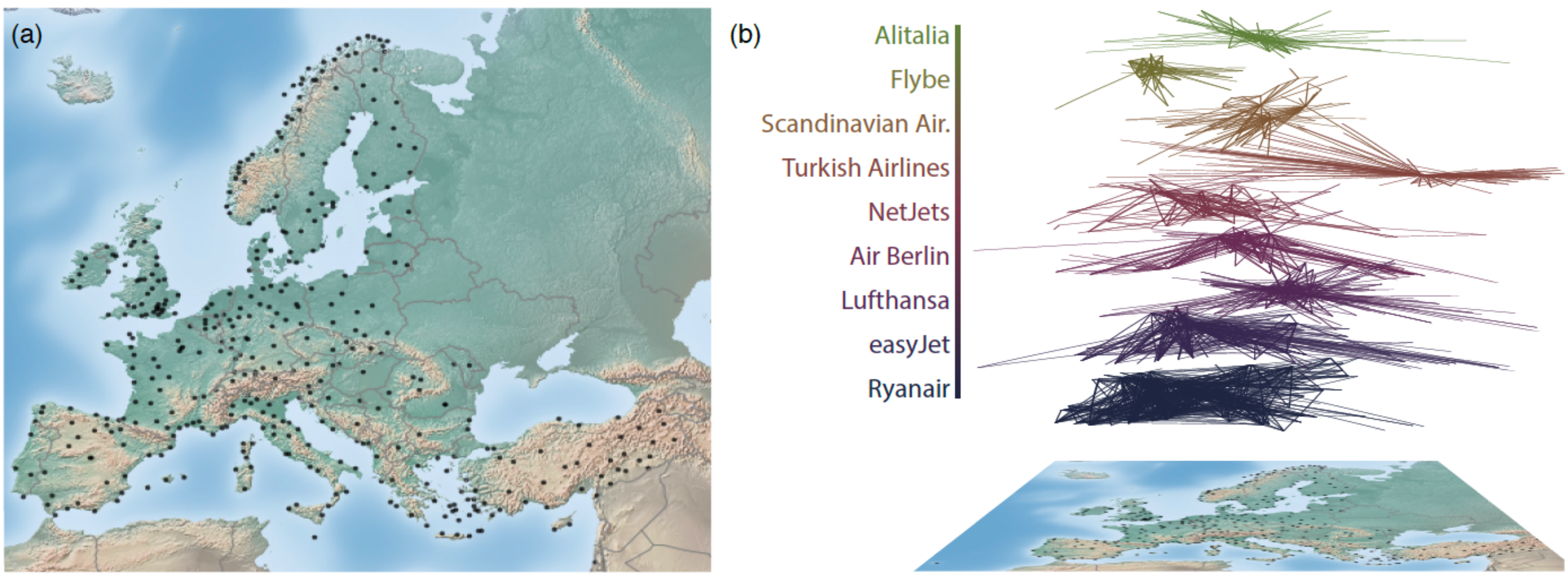
Mathematics PhD Exchange Network

TABLE SM3

Top MNCs (see Definition 3.5 of the main text) for U.S. doctoral programs in the mathematical sciences when the layers' centrality matrices are authority matrices and the interlayer-adjacency matrix is given by (3.5) of the main text. We show results for three choices of (γ, ω).

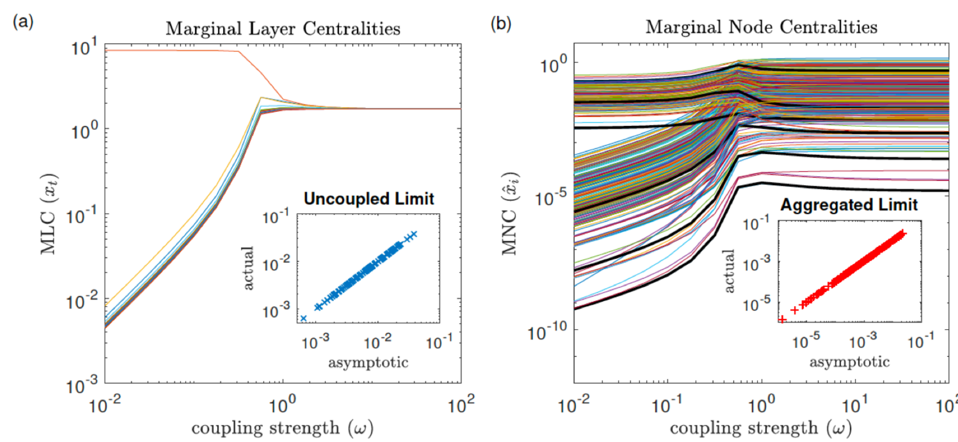
	$(\gamma,\omega) = (10^{-2}, 1)$		$(\gamma,\omega) = (10^{-2}, 10^2)$		$(\gamma,\omega) = (10^{-3}, 10^2)$	
Rank	University	$\hat{x}_i$	University	$\hat{x}_i$	University	$\hat{x}_i$
1	MIT	0.91	MIT	5.28	MIT	3.47
2	U Washington	0.23	UC Berkeley	2.28	UC Berkeley	1.72
3	Boston U	0.15	Stanford	1.84	Stanford	1.28
4	U Michigan	0.12	Princeton	1.42	Harvard	0.97
5	Brown	0.12	Harvard	1.28	Princeton	0.96
6	UCLA	0.111	Cornell	1.23	Cornell	0.89
7	Carnegie Mellon	0.11	UIUC	1.18	UIUC	0.77
8	Purdue	0.11	Washington	1.13	UCLA	0.75
9	USC	0.11	U Michigan	1.12	Wisconsin-Madison	0.74
10	U of Georgia	0.11	UCLA	1.09	U Michigan	0.66

Multiplex Airline Network



Node centralities and layer centralities depend on the interlayer-coupling strength

Layer (t)	Airline name	M_t	$\lambda_1^{(t)}$
1	Lufthansa	244	14.5
2	Ryanair	601	19.3
3	easyJet	307	14.0
4	British Airways	66	6.6
5	Turkish Airlines	118	9.9
6	Air Berlin	184	11.3
7	Air France	69	7.2
8	Scandinavian Air.	110	8.9
9	KLM	62	7.9
10	Alitalia	93	8.8
11	Swiss Int. Air Lines	60	7.3
12	Iberia	35	5.8
13	Norwegian Air Shu.	67	8.1
14	Austrian Airlines	74	8.1
15	Flybe	99	8.5
16	Wizz Air	92	6.5
17	TAP Portugal	53	7.0
18	Brussels Airlines	43	6.6
19	Finnair	42	6.4
20	LOT Polish Air.	55	6.8
21	Vueling Airlines	63	6.8
22	Air Nostrum	69	6.4
23	Air Lingus	108	6.7
24	Germanwings	67	7.4
25	Pegasus Airlines	58	6.7
26	NetJets	180	8.2
27	Transavia Holland	57	6.0
28	Niki	37	4.7
29	SunExpress	67	7.8
30	Aegean Airlines	53	6.5
31	Czech Airlines	41	6.4
32	European Air Trans.	73	6.8
33	Malev Hungarian Air.	34	5.8
34	Air Baltic	45	6.4
35	Wideroe	40	5.6
36	TNT Airways	61	6.2
37	Olympic Air	43	6.2



Conclusions

Conclusions

- Centralities allow one to measure the importance of nodes, edges, and other structures in networks.
- The formalism of multilayer networks allows one to study complicated types of network structures.
  - E.g., temporal networks, multiplex networks, etc.
- There are many different ways to generalize standard network ideas (e.g., centralities) from ordinary graphs to multilayer networks.
- As an example, I discussed eigenvector-based centralities in temporal and multirelational networks.