Laplace Transform and Inverse Laplace Transforms - Numerical Methods, Groups, and Clifford Algebra

Yueyang Shen[1,3], Yupeng Zhang[1,2], Ivo Dinov[1,3]

[1]Statistical Online Computational Resources (SOCR), University of Michigan
[2]Department of Mathematics, University of Wisconsin
[3]Department of Computational Medicine and Bioinformatics, University of Michigan

October 21, 2022

Overview

Integral Transforms

An integral transform is a linear operator:

$$T(f)(u) = \int_{\mathcal{X}} f(x) \underbrace{K(x,u)}_{\text{kernel}} d\mu(x) \tag{1}$$

The form permeates many fields of mathematical modeling. A more familiar form is

$$f(y) = \int f(y \mid \theta) \underbrace{p(\theta)}_{\text{Probability}} d\theta \tag{2}$$

Many formulations in signal processing, machine learning are framed using this framework.

Incarnations of Integral transforms

- Decision Making. Integral transforms find its incarnation in Markov Decision Process theory by modeling the transition dynamics:

$$P_\pi(s'|s) = \sum_a P(s'|s,a)\pi(a|s) \tag{3}$$

- Kernel Machines. Integral transforms and Mercers theorem altogether gives a succinct representation for representing regression function:

$$h(x^{(i)}) = \sum_{j=1}^{n} \alpha_j y^{(j)} K(x^{(i)}, x^{(j)}) \tag{4}$$

- Generative Modeling. Deep fakes and Bayesian statistics and missing data[1]

$$p_{\mathcal{X}}(x) = \int \underbrace{p_g(x \mid z)}_{\text{likelihood}} \quad \underbrace{p_{\mathcal{Z}}(z)}_{\text{sampling prior}} \quad dz \tag{5}$$

[1] Lars Ruthotto and Eldad Haber. "An introduction to deep generative modeling". In: *GAMM-Mitteilungen* 44.2 (2021), e202100008

Laplace transforms - Preliminaries

- The Laplace transform is a **bounded** linear map - by defining $\mathcal{L} : L^2([0,\infty)) \to H^2(\mathbb{C}_+)$, $\|\mathcal{L}f\|_{H^2(\mathbb{C}_+)} = \|f\|_{L^2([0,\infty))}$

Laplace transforms - Preliminaries

- The Laplace transform is a **bounded** linear map - by defining $\mathcal{L} : L^2([0,\infty)) \to H^2(\mathbb{C}_+)$, $\|\mathcal{L}f\|_{H^2}(\mathbb{C}_+) = \|f\|_{L^2([0,\infty))}$
  - The invertibility in this class of function is given by the Bromwhich integral

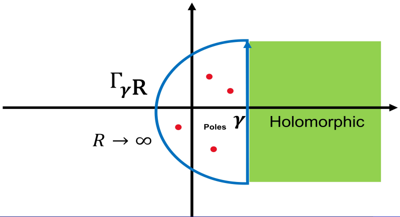
$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F(s)e^{st}\,ds$$

- The Laplace transform is a **bounded** linear map - by defining $\mathcal{L} : L^2([0,\infty)) \to H^2(\mathbb{C}_+)$, $\|\mathcal{L}f\|_{H^2(\mathbb{C}_+)} = \|f\|_{L^2([0,\infty))}$
  - The invertibility in this class of function is given by the Bromwhich integral

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F(s)e^{st}ds$$

  - The general Bromwhich condition is introduced with the decay factor γ such that $e^{-\gamma t}f \in L^2([0,\infty))$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds$$



Laplace transforms - Preliminaries

- The Laplace transform is a **bounded** linear map - by defining $\mathcal{L} : L^2([0,\infty)) \to H^2(\mathbb{C}_+)$, $\|\mathcal{L}f\|_{H^2(\mathbb{C}_+)} = \|f\|_{L^2([0,\infty))}$
  - ▶ The general Bromwhich condition is introduced with the decay factor γ such that $e^{-\gamma t} f \in L^2([0,\infty))$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st} ds$$

▶ The general functional assumption in real life is often intractable. If the function is in the class then we expect to get sensible results for approximation error.

- The Laplace transform is a **bounded** linear map - by defining $\mathcal{L} : L^2([0,\infty)) \to H^2(\mathbb{C}_+)$, $\|\mathcal{L}f\|_{H^2}(\mathbb{C}_+) = \|f\|_{L^2([0,\infty))}$
  - The general Bromwhich condition is introduced with the decay factor γ such that $e^{-\gamma t} f \in L^2([0,\infty))$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st} ds$$

  - The general functional assumption in real life is often intractable. If the function is in the class then we expect to get sensible results for approximation error.
- Laplace transform facilitates the **calculation of moments**. It has a nice connection with moment generating function $X \geq 0$. Using the Laplace transform, we can recover the cumulative distribution function, F_X, of a continuous random variable X.

Laplace transforms - Preliminaries

- The Laplace transform is a **bounded** linear map - by defining $\mathcal{L}: L^2([0,\infty)) \to H^2(\mathbb{C}_+)$, $\|\mathcal{L}f\|_{H^2(\mathbb{C}_+)} = \|f\|_{L^2([0,\infty))}$
  - The general Bromwhich condition is introduced with the decay factor γ such that $e^{-\gamma t} f \in L^2([0,\infty))$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds$$

  - The general functional assumption in real life is often intractable. If the function is in the class then we expect to get sensible results for approximation error.
- Laplace transform facilitates the **calculation of moments**. It has a nice connection with moment generating function $X \geq 0$. Using the Laplace transform, we can recover the cumulative distribution function, F_X, of a continuous random variable X.

$$\mathcal{L}\left\{F_X'\right\}(x) = \underbrace{F(0)}_{0} + s\mathcal{L}\left\{F_X\right\}(x) \Longrightarrow \boxed{F_X(x) = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathbb{E}\left[e^{-sX}\right]\right\}(x) = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\{f_X\}(s)\right\}(x).} \tag{6}$$

Overview

The canonical Fourier analysis can be extended to topological groups by virtue of Peter-Weyl's result[2]

$$\begin{aligned} \text{Forward}: \hat{f}(\rho_l) = [\mathcal{F}_G f]_l &= \int_G f(g)\rho_l(g)dg \\ \text{Backward}: \left[\mathcal{F}_G^{-1}\hat{f}\right]_l &= \sum_l d_{\rho_l} tr\left[\hat{f}(\rho_l)\rho_l(g^{-1})\right] \end{aligned} \tag{7}$$

The case of $SO(2)$: $U(1) = S^1 \cong SO(2)$. 1D complex irreps $\rho_l(g \equiv e^{i\theta}) = e^{il\theta}, l \in \mathbb{Z}$ are the circular harmonics. This is the complex Fourier-Euler basis $\{\sqrt{\frac{1}{2\pi}}e^{in\theta}\}_{n=-\infty}^{\infty}$ for 1D Fourier Series for $L^2([0, 2\pi])$.

$$\mathcal{F}_G^{-1}\hat{f} = f(x) = \sum_{l\in\mathbb{Z}} \frac{1}{2\pi} \underbrace{\int_0^{2\pi} f(\theta)e^{il\theta}d\theta}_{\mathcal{F}_G f} \cdot e^{-ilx} \tag{8}$$

- Limited studies have tried to extend this to groups (locally compact Abelian)-Gelfand transform [3].

[2] Fritz Peter and Hermann Weyl. "Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe". In: *Mathematische Annalen* 97.1 (1927), pp. 737–755

[3] George W Mackey. "The Laplace transform for locally compact Abelian groups". In: *Proceedings of the National Academy of Sciences* 34.4 (1948), pp. 156–162

Clifford Algebra

The main motivation is to define Laplace transform for Clifford spacetime algebra. The main challenge for a closed form solution is the BCH condition suggesting non-commutativity:

$$
\begin{aligned}
e^{M_1}e^{M_2} &= e^{M_1+M_2+\frac{1}{2}[M_1,M_2]+\frac{1}{12}([M_1,[M_1,M_2]]+[M_2,[M_2,M_1]])+\cdots} \\
e^{M_1}e^{M_2} &\neq e^{M_1+M_2} \text{ for } [M_1, M_2] \neq 0
\end{aligned} \tag{9}
$$

We may formulate this in a more general framework of geometric algebra (Clifford algebra), where

1. $\mathrm{Cl}_{1,3}(\mathbb{R})$ correspond to the spacetime algebra with metric signature $(+,-,-,-)$
2. $\mathrm{Cl}_{0,2}(\mathbb{R})$. Quarternion ($\mathbb{H} \cong \mathrm{Cl}_{0,2}(\mathbb{R})$)
3. $\mathrm{Cl}_{0,1}(\mathbb{R})$ complex numbers ($\mathbb{C} \cong \mathrm{Cl}_{0,1}(\mathbb{R})$)
4. $\mathrm{Cl}_{0,0}(\mathbb{R})$ real numbers ($\mathbb{R} \cong \mathrm{Cl}_{0,0}(\mathbb{R})$)

Clifford-Fourier Transforms

- The Case for $\mathrm{Cl}_{2,0}(\mathbb{R})$. The vector space G^2 of the algebra contains the basis $\{1, e_1, e_2, e_1 e_2\}$. The peculiarity is that $(e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -e_1 e_2 e_2 e_1 = -1$. The bivector $e_1 e_2$ is associated with a pseudoscalar (correspond to highest grade basis) $i_2 = e_1 e_2, i_2^2 = -1$. The pseudoscalar gives an alternative expression for the basis decomposition $\boldsymbol{f}(x) = f_0 + f_1 e_1 + f_2 e_2 + f_{12} e_{12} = 1(f_0(x) + f_{12}(x) i_2) + e_1(f_1(x) + f_2(x) i_2)$. In the algebraic constraints, i_2 behaves like the imaginary number i hence the 2D Clifford Fourier transform operates on spinor $(f_0(x) + f_{12}(x) i_2)$ and vector $(f_1(x) + f_2(x) i_2)$ component separately is naturally defined as [4]

$$\hat{\boldsymbol{f}}(\xi) = \mathcal{F}\{\boldsymbol{f}\}(\xi) = \int_{\mathbb{R}^2} \boldsymbol{f}(x) e^{-2\pi i_2 \langle x, \xi \rangle} dx, \quad \forall \xi \in \mathbb{R}^2 \tag{10}$$

$$\boldsymbol{f}(x) = \mathcal{F}^{-1}\{\mathcal{F}\{\boldsymbol{f}\}\}(x) = \int_{\mathbb{R}^2} \hat{\boldsymbol{f}}(\xi) e^{2\pi i_2 \langle x, \xi \rangle} d\xi, \quad \forall x \in \mathbb{R}^2 \tag{11}$$

[4] Johannes Brandstetter et al. "Clifford neural layers for pde modeling". In: *arXiv preprint arXiv:2209.04934* (2022)

Clifford Fourier Transform

- The Case for $\text{Cl}_{0,2}(\mathbb{R})$. Quarternion ($\mathbb{H} \cong \text{Cl}_{0,2}(\mathbb{R})$)The Clifford Fourier transform [5]

$$\mathcal{F}^{cl}\{f\}(\underbrace{u_1, u_2}_{\boldsymbol{u}}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) e^{-2\pi e_1 u_1 x_1} e^{-2\pi e_2 u_2 x_2} d\boldsymbol{x} \tag{12}$$

$$f(\boldsymbol{x}) = (\mathcal{F}^{-1})^{cl}\{\hat{f}\}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{u}) e^{2\pi e_2 x_2 u_2} e^{2\pi e_1 x_1 u_1} d\boldsymbol{u} \tag{13}$$

The more specialized transform for quaternions is to sandwiching the function using exponentials

$$\mathcal{F}^{q}\{f\}(\underbrace{u_1, u_2}_{\boldsymbol{u}}) = \int_{\mathbb{R}^2} e^{-2\pi e_1 u_1 x_1} f(\boldsymbol{x}) e^{-2\pi e_2 u_2 x_2} d\boldsymbol{x} \tag{14}$$

$$f(\boldsymbol{x}) = (\mathcal{F}^{-1})^{q}\{\hat{f}\}(\boldsymbol{x}) = \int_{\mathbb{R}^2} e^{2\pi e_1 x_1 u_1} \hat{f}(\boldsymbol{u}) e^{2\pi e_2 x_2 u_2} d\boldsymbol{u} \tag{15}$$

where $e_1 = \hat{\boldsymbol{i}}, e_2 = \hat{\boldsymbol{j}}, e_1 e_2 = \hat{\boldsymbol{i}}\hat{\boldsymbol{j}} = \hat{\boldsymbol{k}}$.

[5] Eckhard Hitzer and Stephen J Sangwine. *Quaternion and Clifford Fourier transforms and wavelets*. Springer, 2013

Clifford Fourier Transform

- The Case for spacetime algebra $\mathrm{Cl}_{3,1}(\mathbb{R})$. We can extend the defintion to complex signatures and obtaining transforms for spacetime algebras utilizing the pseudoscalars and quarternion fourier transform definition with east coast metric signature $(+,+,+,-)$ [6]

$$\hat{f}(\boldsymbol{u}) = \mathcal{F}\{f\}(\boldsymbol{u}) = \int_{\mathbb{R}^{3,1}} e^{-2\pi e_0 ts} f(\boldsymbol{x}) e^{-2\pi i_3 \langle x,u \rangle} d^4\boldsymbol{x} \tag{16}$$

$$f(\boldsymbol{x}) = (\mathcal{F}^{-1})\{\hat{f}\}(\boldsymbol{x}) = \int_{\mathbb{R}^{3,1}} e^{2\pi e_0 ts} \hat{f}(\boldsymbol{u}) e^{2\pi i_3 \langle x,u \rangle} d^4\boldsymbol{u} \tag{17}$$

where $f : \mathbb{R}^{3,1} \to \mathrm{Cl}_{3,1}(\mathbb{R})$, and i_3 is the pseudo-scalar in $\mathrm{Cl}_{3,0}(\mathbb{R})$, where the spacetime vectors and spacetime frequency are defined by $\boldsymbol{x} = te_0 + x, x = x_1 e_1 + x_2 e_2 + x_3 e_3$ and $\boldsymbol{u} = se_0 + u, u = u_1 e_1 + u_2 e_2 + u_3 e_3$ respectively.

[6] Eckhard Hitzer and Stephen J Sangwine. *Quaternion and Clifford Fourier transforms and wavelets.* Springer, 2013

Overview

Basis Approximation

Basis Function	Transformation formula	Orthogonality Condition
Sine Basis	$\mathcal{L}(\sin(wt)H(t))(z) = \frac{w}{z^2+w^2}$	$\int_0^{2\pi} \sin(nt)\sin(mt)dt = \pi\delta_{m,n}$
Cosine Basis	$\mathcal{L}(\cos(wt)H(t))(z) = \frac{w}{z^2+w^2}$	$\int_0^{2\pi} \cos(nt)\cos(mt)dt = \pi\delta_{m,n}$
Bessel Basis	$\mathcal{L}(J_n(at)H(t))(s) = \frac{(\sqrt{s^2+a^2}-s)^n}{a^n\sqrt{a^2+s^2}}$	$\int_0^1 tJ_n(at)J_n(bt)dt = \frac{1}{2}J_n'(a)^2\delta_{a,b}$
Laguerre Basis[27]	$\mathcal{L}(L_n^{(\alpha)}(t)H(t))(z)=\frac{\Gamma(1+\alpha+n)}{\Gamma(1+\alpha)n!s}M(-n,1+\alpha;\frac{1}{s})$	$\int_0^\infty t^\alpha e^{-t}L_n^{(\alpha)}(t)L_m^{(\alpha)}(t)dt=\frac{\Gamma(1+\alpha+n)\delta_{m,n}}{n!}$
Legendre Basis[28]	$\mathcal{L}(P_n(t)H(t))(s)=\frac{1}{2}\sqrt{\pi}\Big(\sqrt{\frac{2}{s}}I_{-n-1/2}(s)+$ $(-1/2s)^{\lfloor\frac{\lvert n\rvert+2}{2}\rfloor-\lceil\frac{\lvert n\rvert}{2}\rceil}{}_1F_2(1;\frac{1}{2}n+2-\frac{1}{2}\lceil\frac{\lvert n\rvert}{2}\rceil),$ $1+\frac{1}{2}(\lceil\frac{\lvert n+1\rvert}{2}\rceil-\lfloor\frac{\lvert n+1\rvert+2}{2}\rfloor)-\frac{1}{2}n;\frac{1}{4}s^2)\Big)$	$\int_{-1}^1 P_n(t)P_m(t)dt = \frac{2}{2n+1}\delta_{m,n}$
Hermite Basis	$\mathcal{L}(H_n(t)H(t))(z)=2^n\frac{\Gamma(1+n)}{s^{1+n}}{}_2F_2(-\frac{n}{2},\frac{1-n}{2};\frac{-n}{2},\frac{1-n}{2};\frac{s^2}{4})$	$\int_{-\infty}^\infty H_m(x)H_n(x)e^{-x}dx = \sqrt{\pi}2^n n!\delta m,n$
Chebyshev Basis[29]	$\mathcal{L}(H_n(t)x^\alpha T_n(1-2x))(z)=\frac{\Gamma(1+\alpha)}{z^{1+\alpha}}{}_3F_1(-n,n,1+\alpha;1/2;1/z)$	$\int_{-1}^1 \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \frac{1}{2}^{\delta n,0}\pi\delta_{n,m}$

Table 1: The $M(\cdot)$ is the confluent hypergeometric function, $I_n(\cdot)$ is the modified Bessel function of type n

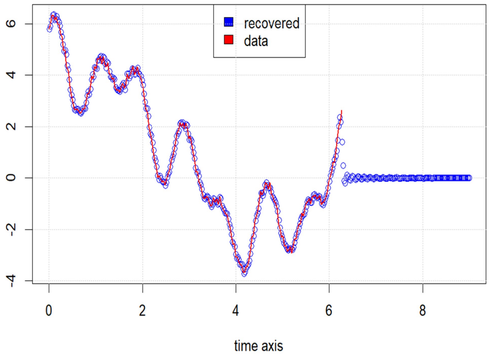
Algorithm 1 Randomized ILT

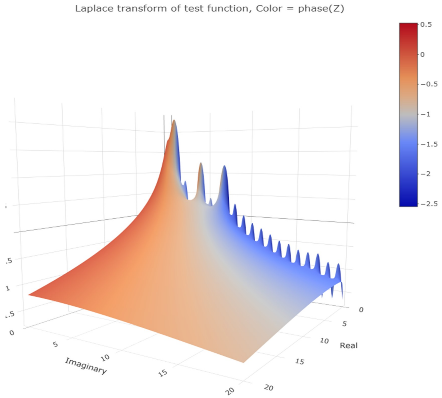
$N_1, N_2 \leftarrow$ *prior_estimate* is the partition size
$itn \leftarrow g(N_1, N_2, \textit{prior\_estimate})$ is the number of attempts (*iterations*)
for $1 \leq k \leq itn$ **do**
    $\mathbf{p}^k \leftarrow$ Random N_1 size partition on interval $(0, 2\pi)$, according to distribution P.
    $\mathbf{b} = (b_i) \leftarrow$ Random N_2 points (z_i, b_i) from the dataset S
    $\mathbf{A} \leftarrow (a_{ij} = \frac{1}{-z_i}(\exp(-z_i p_j) - \exp(-z_i p_{j-1}))$, note that $\mathbf{A}$ is the matrix computing LT of a quantized piecewise constant function
    $\mathbf{u}^{pk} = \mathbf{A}^{-1}\mathbf{b}$
    $f^{pk} \leftarrow \mathbf{u}^{pk}$ as a piecewise constant function
end for
$f \leftarrow \frac{1}{itn} \sum_k f^{pk}$

Unfortunately, the entries of our LT matrix A are far from independent and A may not be Hermitian. Hence, there is little guarantee that direct applications of random matrix theory may ensure reasonable approximations of the smallest singular value of the LT matrix. However, we are able to perform empirical evaluation on this method.

Empirical Results - LT

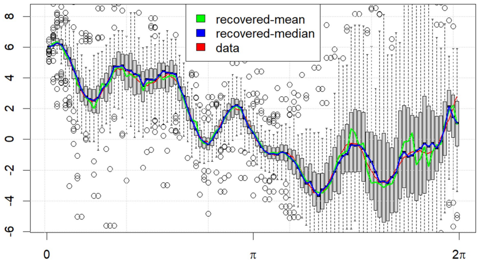
$$f(x) = 2\sin(x) + \cos(4x) + \sin(7x + 0.5) + (x - 3)(x - 5) * 0.3 + \varepsilon(x)$$

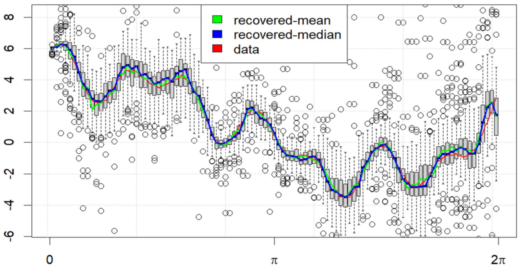




Empirical Result - ILT

$$f(x) = 2\sin(x) + \cos(4x) + \sin(7x + 0.5) + (x - 3)(x - 5) * 0.3 + \varepsilon(x)$$





Overview

SOCR technologies

- TCIU package: https://cran.r-project.org/web/packages/TCIU/index.html
- Github link: https://github.com/SOCR/TCIU
- SOCR link: https://www.socr.umich.edu/
- TCIU tutorials: https://www.socr.umich.edu/TCIU/

References

[1] Johannes Brandstetter et al. "Clifford neural layers for pde modeling". In: *arXiv preprint arXiv:2209.04934* (2022).

[2] Eckhard Hitzer and Stephen J Sangwine. *Quaternion and Clifford Fourier transforms and wavelets*. Springer, 2013.

[3] George W Mackey. "The Laplace transform for locally compact Abelian groups". In: *Proceedings of the National Academy of Sciences* 34.4 (1948), pp. 156–162.

[4] Fritz Peter and Hermann Weyl. "Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe". In: *Mathematische Annalen* 97.1 (1927), pp. 737–755.

[5] Lars Ruthotto and Eldad Haber. "An introduction to deep generative modeling". In: *GAMM-Mitteilungen* 44.2 (2021), e202100008.