Non-backtracking Spectra of Random Hypergraphs and Community Detection

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Tensor Representation, Completion, Modeling and Analytics of Complex Data

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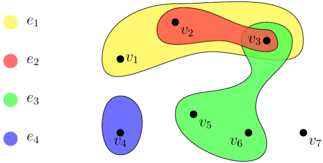
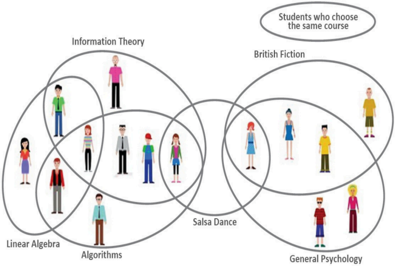
Joint work with Ludovic Stephan (EPFL)

Hypergraph

- $G = (V, H)$, V: vertex set, H: hyperedge set.

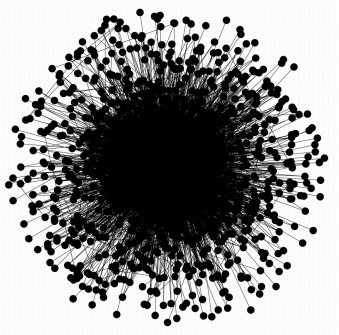


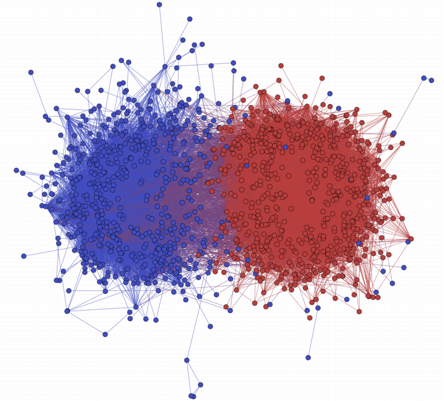


Ravindran '15

- Higher-order networks: co-authorship, chat group, protein interaction

Community detection





Political blogs data from Adamic-Glance '05. Figure from Abbe '18

Community detection on random graphs

- Consider a (unknown) partition of n vertices into two *communities* of size $n/2$. Generate edges within each community with probability p. Generate edges across communities with probability $q < p$.
- **Stochastic block model** $\mathcal{G}(n, p, q)$. Holland et al. '83.

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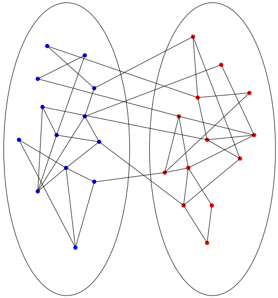
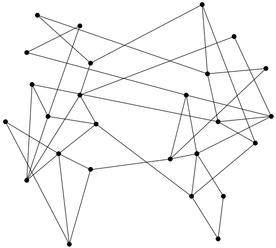
- **Stochastic block model** $\mathcal{G}(n,p,q)$. Holland et al. '83.
- Task: observe a graph $G \sim \mathcal{G}(n,p,q)$, find the unknown partition with high probability (efficiently and accurately).





Spectral method on the adjacency matrix

- Adjacency matrix A: symmetric, A_{ij} is independent Bernoulli for $i < j$.
- $\mathbb{E}A = \left[\begin{array}{cc|cc} p & p & q & q \\ p & p & q & q \\ \hline q & q & p & p \\ q & q & p & p \end{array}\right], \quad \lambda_1(\mathbb{E}A) = \frac{(p+q)n}{2}, \quad \lambda_2(\mathbb{E}A) = \frac{(p-q)n}{2}.$
- $v_1(\mathbb{E}A) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2(\mathbb{E}A) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$
- $A = \mathbb{E}A + (A - \mathbb{E}A)$, low rank + noise.

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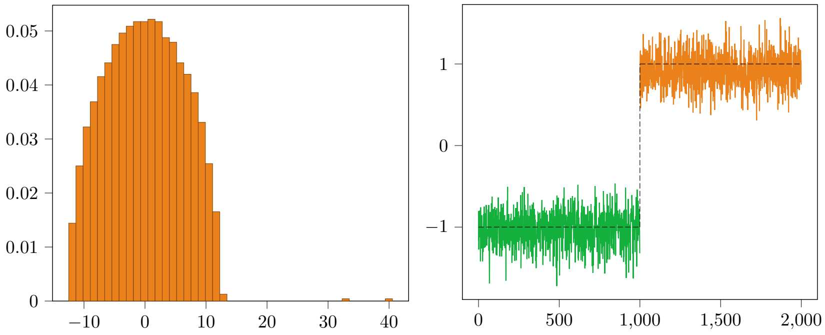
- $A = \mathbb{E}A + (A - \mathbb{E}A)$, low rank + noise.
- If A is concentrated around $\mathbb{E}A$, then $v_2(A) \approx v_2(\mathbb{E}(A))$.
- Spectral method: observe A, compute $v_2(A)$, use the signs of the entries in $v_2(A)$ to recover the community.

Feige–Ofek '05, Lei–Rinaldo '13, Le–Levina–Vershynin '16, Benaych Georges–Bordenave–Knowles '17, Latala–van Handel–Youssef '17, Alt–Ducatez–Knowles '19, Tikhomirov–Youssef '19

Spectral method on A: dense regime



Exact recovery when $p = \frac{a \log n}{n}, q = \frac{b \log n}{n}$ and $(\sqrt{a} - \sqrt{b})^2 \geq 2$.

[Abbe-Bandeira-Hall '15, Mossel-Neeman-Sly '16, Abbe-Fan-Wang-Zhong '20].

Sparse SBMs

- Two communities of equal size. $\sigma : [n] \to \{-1, 1\}$.
- Bounded expected degrees: $p = \frac{a}{n}, q = \frac{b}{n}$. Impossible to recover σ exactly.
- Detection of σ is possible (strictly better than random guessing) if and only if $(a-b)^2 > 2(a+b)$ (Kesten-Stigum threshold).

Decelle-Krzakala-Moore-Zdeborová '11, Mossel-Neeman-Sly '15, '18, Massoulié '14, Bordenave-Lelarge-Massoulié '18.

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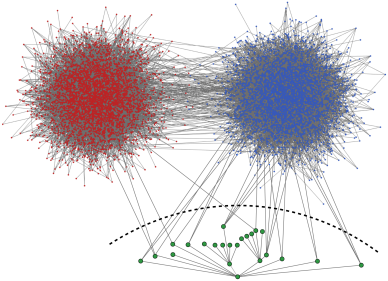
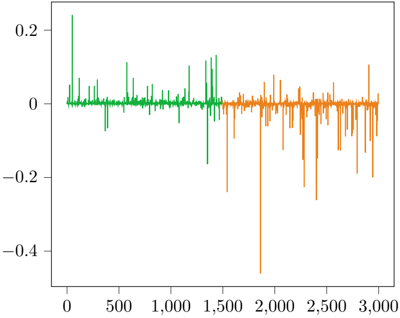




Top eigenvectors of A are localized on high degree vertices.

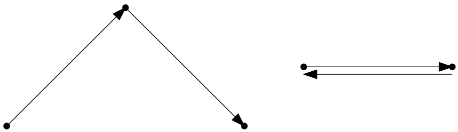
Non-backtracking operator

The set of oriented edges:

$$\vec{E} = \{u \to v : \{u, v\} \in E\}.$$

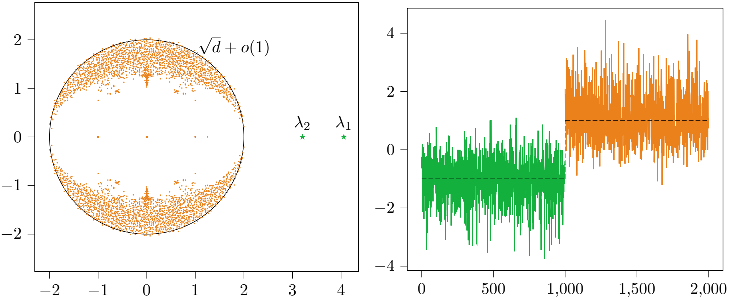
$|\vec{E}| = 2|E|$. The non-backtracking operator B is defined on $\vec{E}$.
For $u \to v, x \to y \in \vec{E}$,

$$B_{u\to v, x\to y} = \mathbf{1}_{v=x}\mathbf{1}_{u\neq y}.$$



Bordenave-Collins '19, Bordenave '20, Brito-Dumitriu-Harris '22, Benaych-Georges, Bordenave, Knowles '21, Stephan-Massoulié '20, Bordenave-Coste-Nadakuditi '20,. . .

Spectral method on B



[Bordenave, Lelarge, Massoulié '18] Let $p = \frac{a}{n}, q = \frac{b}{n}$. If $(a-b)^2 > 2(a+b)$, then with high probability,

$$\lambda_1(B) = \frac{a+b}{2} + o(1), \quad \lambda_2(B) = \frac{a-b}{2} + o(1), \quad |\lambda_3(B)| \leq \sqrt{\frac{a+b}{2}} + o(1).$$

The second eigenvector of B can be used to detect σ.

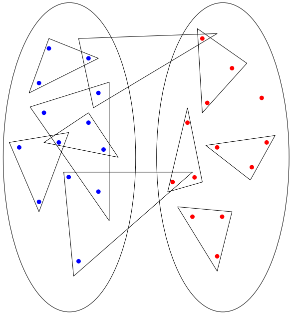
A fails but B works (optimally)!

Hypergraph stochastic block model (HSBM)

G is *q-uniform* if each hyperedge has size q.

- Community assignment $\sigma : [n] \to \{-1, +1\}$.
- Each hyperedge $e = \{v_1, \ldots, v_q\}$ appears independently with probability

$$\mathbb{P}(e \in H) = \begin{cases} c_{\mathrm{in}} & \text{if } \sigma_{v_1} = \cdots = \sigma_{v_q} \\ c_{\mathrm{out}} & \text{otherwise.} \end{cases}$$



Task: observe G, construct a label estimator $\hat{\sigma} \in \{-1, +1\}^n$ correlated with the true σ. Ghoshdastidar-Dukkipati '14, Chien-Lin-Wang '18, Kim-Bandeira-Goemans '18, Ahn-Lee-Suh '18, ... when expected degree (expected number of hyperedges containing a vertex) $d \to \infty$.

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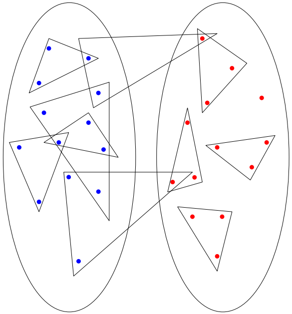

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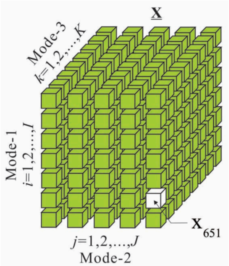
- Detection: [Angelini-Caltagirone-Krzakala-Zdeborová '15] conjectured a phase transition when $c_{\rm in} = \frac{a}{\binom{n}{q-1}}, c_{\rm out} = \frac{b}{\binom{n}{q-1}}$.
- Provable spectral method in the bounded expected degree regime?

Tensor

The **adjacency tensor** T: sparse random tensor of order q with n^q many entries. $T_{i_1,\ldots,i_q} = 1$ if $\{i_1, \ldots, i_q\}$ is a hyperedge.



Most tensor problems are NP-hard (Hillar-Lim '13): rank, spectral norm, best low-rank approximation,. . .

Tucker decomposition: Ghoshdastidar-Dukkipat '17, Ke-Shi-Xia '20 for $d = \omega(\log^2 n)$.

Adjacency matrix

Define the **adjacency matrix** of G as

$$A_{ij} := \{\text{number of hyperedges containing } i, j\}.$$

Spectral method on A fails when $d = O(1)$.

Does the non-backtracking method work for random hypergraphs?
[Stephan-Z. '22]: Yes, and efficient for a more general HSBM model.

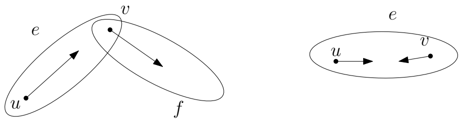
Non-backtracking operator for hypergraphs

For a given hypergraph $G = (V, H)$, let $\vec{H}$ be the *oriented hyperedge* in G such that

$$\vec{H} = \{(v, e) : v \in e \cap V, e \in H\}, \quad |\vec{H}| = q|H|.$$

B: a matrix indexed by $\vec{H}$ such that

$$B_{(u \to e),(v \to f)} = \begin{cases} 1 & \text{if } v \in e \setminus \{u\}, f \neq e, \\ 0 & \text{otherwise.} \end{cases}$$



Storm '06, Angelini-Caltagirone-Krzakala-Zdeborová '15, Dumitriu-Z. '21.

Generate an HSBM from a probability tensor

- An order-q *symmetric probability tensor* $\mathbf{P} \in \mathbb{R}^{r^q}$ and $\sigma : [n] \to [r]$.
- Each hyperedge of size q is included in H with probability

$$\mathbb{P}(e \in H) = \frac{p_{\underline{\sigma}(e)}}{\binom{n}{q-1}},$$

  where $\underline{\sigma}(e) = \underline{\sigma}(\{v_1, \dots, v_q\}) := (\sigma(v_1), \dots, \sigma(v_q))$.
- The proportion of each community is $\pi_i, i \in [r]$. Assume each vertex has the same expected degree d.

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The nonzero eigenvalues of $\mathbb{E}A$ are given by

$$|\mu_r| \le \cdots \le |\mu_2| \le \mu_1 = d.$$

Denote by r_0 the number of informative eigenvalues, or equivalently

$$(q-1)\mu_{r_0+1}^2 \le d < (q-1)\mu_{r_0}^2.$$

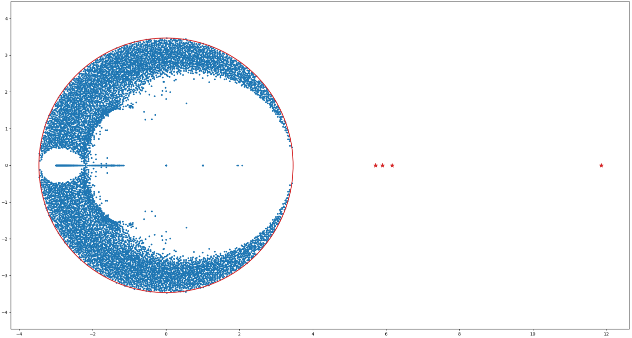
Same as the generalized KS threshold conjectured in [Angelini et al. '15].

Spectrum of B

Theorem (Stephan-Z., FOCS 22)

Let G be a hypergraph generated according to the HSBM with m hyperedges, and B be its non-backtracking matrix and $|\lambda_1(B)| \geq |\lambda_2(B)| \geq \cdots \geq |\lambda_{qm}(B)|$. Then with high probability:

1. *For any $i \in [r_0]$, $\lambda_i(B) = (q-1)\mu_i + o(1)$.*
2. *For all $r_0 < i \leq qm$, $|\lambda_i(B)| \leq \sqrt{(q-1)d} + o(1)$.*



$n = 6000$, $r = 4$. The parameters c_{in} and c_{out} have been chosen so that $d = 4$ and $\mu_2 = 2$.

Dimension reduction

B has size $q|H| \sim qdn$, but qd could be a large constant. We also need to embed eigenvectors into $\mathbb{R}^n$. Define the $2n \times 2n$ matrix $\tilde{B}$ as

$$\tilde{B} = \begin{pmatrix} 0 & (D-I) \\ -(q-1)I & A-(q-2)I \end{pmatrix},$$

where D is the diagonal *degree matrix* with $D_{ii} = (q-1)^{-1} \sum_j A_{ij}$.

Lemma (Stephan-Z., FOCS 22)

The following Ihara-Bass formula holds:

$$\det(B - zI) = (z-1)^{(q-1)|H|-n} (z+(q-1))^{|H|-n} \det\left(\tilde{B} - zI\right).$$

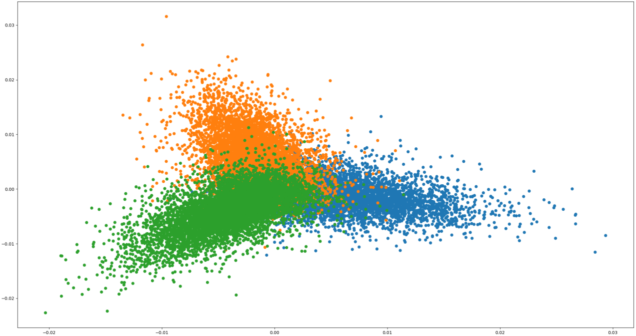
B and $\tilde{B}$ share the same non-trivial eigenvalues.

Eigenvector overlaps

Theorem (Stephan-Z., FOCS 22)

For $i \in [r_0]$, let $\tilde{u}_i$ be the last n entries of the i-th eigenvector of $\tilde{B}$ with $\|\tilde{u}_i\| = 1$. Then with high probability, there exists a unit eigenvector $\tilde{\phi}_i$ of $\mathbb{E}A$ associated to μ_i such that

$$\langle \tilde{u}_i, \tilde{\phi}_i \rangle = \sqrt{\frac{1-\tau_i}{1+\frac{q-2}{(q-1)\mu_i}}} + o(1) \quad \textit{where } \tau_i = \frac{d}{(q-1)\mu_i^2}.$$



Scatter plot of the second and third eigenvector of $\tilde{B}$ under the symmetric HSBM with $q = 4$, $r = 3$.

Conclusions

- Community detection for sparse random hypergraphs can be reduced to an eigenvector problem of a $2n \times 2n$ non-normal matrix constructed from A (**without** the knowledge of T), and it works down to the conjectured generalized KS threshold.
- Ihara-Bass formula and non-backtracking method for non-uniform hypergraphs [Chodrow-Eikmeier-Haddock '22].