Complex-time Representation of Repeated Measurement Longitudinal Data & Space-kime Analytics

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Joint work with Milen V. Velev (BTU) & Yueyang Shen (UM)

Based on the book "*Data Science: Time Complexity, Inferential Uncertainty & Spacekime Analytics*"

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Outline

- ❑ Complex-Time (*kime*), Rationale & Applications
- ❑ Solutions of ultrahyperbolic wave equations
- ❑ Open Spacekime Problems
- ❑ Bayesian Formulation of Spacekime Inference
- ❑ Resources & Live Demo Links

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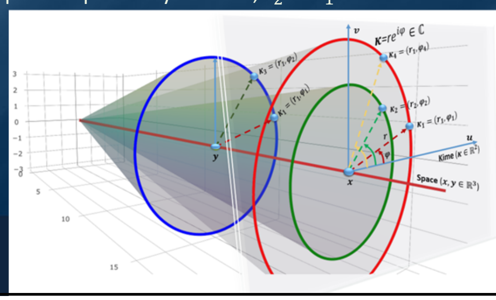
Complex-Time (*Kime*)

- At a given spatial location, $\boldsymbol{x}$, complex time (*kime*) is defined by $\kappa = re^{i\varphi} \in \mathbb{C}$, where:
  - the <u>magnitude</u> represents the longitudinal events order ($r > 0$) and characterizes the longitudinal displacement in time, and
  - event <u>phase</u> ($-\pi \le \varphi < \pi$) is an angular displacement, event direction, or random sampling index
- There are multiple alternative parametrizations of kime in the complex plane
- Space-kime manifold is $\mathbb{R}^3 \times \mathbb{C}$:
  - $(\boldsymbol{x}, k_1)$ and $(\boldsymbol{x}, k_4)$ have the same spacetime representation, but different spacekime coordinates,
  - $(\boldsymbol{x}, k_1)$ and $(\boldsymbol{y}, k_1)$ share the same kime, but represent different spatial locations,
  - $(\boldsymbol{x}, k_2)$ and $(\boldsymbol{x}, k_3)$ have the same spatial-locations and kime-directions, but appear sequentially in order, $r_2 < r_1$.



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Rationale for *Time* → *Kime* Extension

- <u>**Math**</u> – $Time$ is a special case of *kime*, $\kappa = |\kappa|e^{i\varphi}$ where $\varphi = 0$ (nil-phase)
  - $\mathbb{R}^+$ is algebraically a *multiplicative* (algebraic) group, with multiplicative unity (identity) = 1, multiplicative inverses $t^{-1} = \frac{1}{t}$, and associativity law $t_1 \times (t_2 \times t_3) = (t_1 \times t_2) \times t_3$
  - The $time$ domain ($\mathbb{R}^+$) is **not** a complete *algebraic field* w.r.t. $(+\,,\ \times)$:
    - Additive unity (0), element additive inverse $(-t)$: $t + (-t) = 0$; is outside $\mathbb{R}^+$ (time-domain)
    - $x^2 + 1 = 0$ has no solutions in time (or in $\mathbb{R}$) ….

$$\text{Group}(\times) \subseteq Ring\left(\underbrace{\overbrace{(+,*)}^{\text{Compatible operations}}}_{\text{associative \& distributive}}\right) \subseteq Field\ \overbrace{(+,*)}^{\text{Group}(+)}$$

  - Classical time ($\mathbb{R}^+$) is a *positive cone* over the field of the real numbers ($\mathbb{R}$)
  - **Time ($\mathbb{R}^+$) forms a subgroup of the multiplicative group of the reals**
  - **Whereas <u>kime</u> ($\mathbb{C}$) is an algebraically *closed prime field* that naturally extends time**
  - *Time* is ordered but *kime* is not! Yet, the kime magnitude preserves the intrinsic time order
  - Kime ($\mathbb{C}$) represents the smallest natural extension of time, as a complete filed that agrees with time
  - The $time$ group is closed under addition, multiplication, and division (but not subtraction). It has the topology of $\mathbb{R}$ and the structure of a multiplicative topological group ≡ additive topological semigroup
- <u>**Physics**</u> –
  - Problem of time … (DOI 10.1007/978-3-319-58848-3)
  - $\mathbb{R}$ and $\mathbb{C}$ Hilbert-space quantum theories make different predictions (DOI: 10.1038/s41586-021-04160-4)
- <u>**AI/Data Science**</u> – Random IID sampling, Bayesian reps, tensor modeling of $\mathbb{C}$ kimesurfaces, novel analytics

Dinov & Velev (2021)

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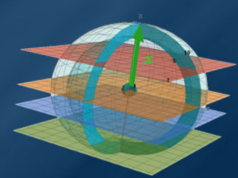
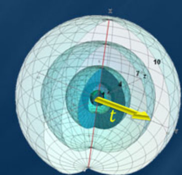
Uncertainty in 5D Spacekime

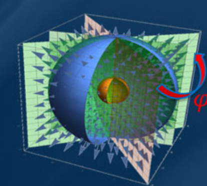
- 5D Space-Time-Matter Consortium showed that in a 5D universe with an extra time dimension, particle spacetime motion may be slightly modified by an extra force to produce a correlation between the momentum and position similar to the uncertainty relation in quantum mechanics
- One component of this additional force is parallel to the 4-velocity and explains the intrinsic Heisenberg uncertainty relation in the lower 4D spacetime embedding
- We can represent classical 4D spacetime Heisenberg uncertainty as a reduction of Einstein-like 5D deterministic dynamics
- (Paul Wesson) "*Heisenberg was right in 4D, because Einstein was right in 5D*"
- D-dimensional "generating" space *foliated* by a family of $(D-1)$ hypersurfaces



Space (x) Foliation of Spacekime



(Radial, t) Time-Foliation of Spacekime



(Angular, φ) Phase-Foliation of *Kime*

Wesson (2004, 2010) | Wesson & Overduin (2018) | Dinov & Velev (2021)

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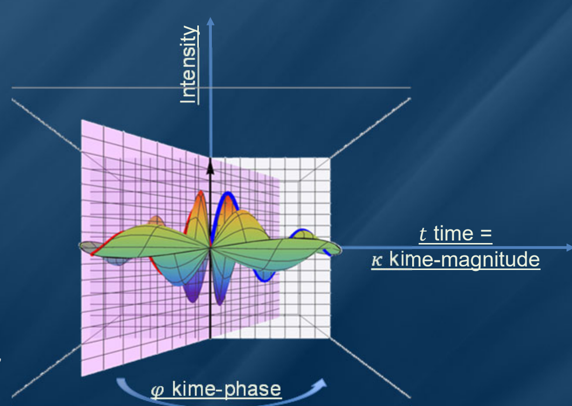
Spacekime Analytics: Kime-series = Surfaces (not curves)

In the 5D spacekime manifold, time-series curves extend to kime-series, i.e., surfaces parameterized by kime-magnitude (t) and the kime-phase (φ)

Kime-phase aggregating operators that can be used to transform standard time-series curves to spacekime kime-surfaces, which can be modeled, interpreted, and predicted using advanced spacekime analytics



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Ultrahyperbolic Wave Equation – Cauchy Initial Data

- **<u>Nonlocal constraints</u>** yield the existence, uniqueness & stability of local and global solutions to the ultrahyperbolic wave equation under Cauchy initial data …

$$\underbrace{\sum_{i=1}^{d_s} \partial_{x_i}^2 u \equiv \Delta_{\boldsymbol{x}} u(\boldsymbol{x},\boldsymbol{\kappa})}_{\text{spatial Laplacian}} = \underbrace{\Delta_{\boldsymbol{\kappa}} u(\boldsymbol{x},\boldsymbol{\kappa}) \equiv \sum_{i=1}^{d_t} \partial_{\kappa_i}^2 u}_{\text{temporal Laplacian}}, \quad \underbrace{\left| \begin{array}{l} u_o = u\big(\underbrace{\boldsymbol{x}}_{x\in D_s}, \underbrace{0,\boldsymbol{\kappa}_{-1}}_{\boldsymbol{\kappa}\in D_t}\big) = f(\boldsymbol{x},\boldsymbol{\kappa}_{-1}) \\ u_1 = \partial_{\kappa_1} u(\boldsymbol{x},0,\boldsymbol{\kappa}_{-1}) = g(\boldsymbol{x},\boldsymbol{\kappa}_{-1}) \end{array} \right.}_{\text{initial conditions (Cauchy Data)}}$$

where $\boldsymbol{x} = (x_1, x_2, \dots, x_{d_s}) \in \mathbb{R}^{d_s}$ and $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \dots, \kappa_{d_t}) \in \mathbb{R}^{d_t}$ are the Cartesian coordinates in the d_s space and d_t time dims.

Stable local solution over a Fourier frequency region defined by <u>nonlocal constraints</u> $|\boldsymbol{\xi}| \geq |\boldsymbol{\eta}_{-1}|$:

$$\hat{u}\big(\boldsymbol{\xi}, \underbrace{\kappa_1, \boldsymbol{\eta}_{-1}}_{\boldsymbol{\eta}}\big) = \cos\left(2\pi\,\kappa_1\sqrt{|\boldsymbol{\xi}|^2 - |\boldsymbol{\eta}_{-1}|^2}\right) \underbrace{\hat{u}_o(\boldsymbol{\xi},\boldsymbol{\eta}_{-1})}_{c_1} + \sin\left(2\pi\,\kappa_1\sqrt{|\boldsymbol{\xi}|^2 - |\boldsymbol{\eta}_{-1}|^2}\right) \underbrace{\frac{\hat{u}_1(\boldsymbol{\xi},\boldsymbol{\eta}_{-1})}{2\pi\sqrt{|\boldsymbol{\xi}|^2 - |\boldsymbol{\eta}_{-1}|^2}}}_{c_2},$$

where $\mathcal{F}\begin{pmatrix} u_o \\ u_1 \end{pmatrix} = \begin{pmatrix} \hat{u}_o \\ \hat{u}_1 \end{pmatrix} = \begin{pmatrix} \hat{u}_o(\boldsymbol{\xi},\boldsymbol{\eta}_{-1}) \\ \hat{u}_1(\boldsymbol{\xi},\boldsymbol{\eta}_{-1}) \end{pmatrix} = \begin{pmatrix} \hat{u}(\boldsymbol{\xi},\boldsymbol{\eta}_{-1}) \\ \partial_{\kappa_1}\hat{u}(\boldsymbol{\xi},\boldsymbol{\eta}_{-1}) \end{pmatrix}$.

$$u\big(\boldsymbol{x}, \underbrace{\kappa_1, \boldsymbol{\kappa}_{-1}}_{\boldsymbol{\kappa}}\big) = \mathcal{F}^{-1}(\hat{u})(\boldsymbol{x},\boldsymbol{\kappa}) = \int_{\hat{D}_s \times \hat{D}_{t_{-1}}} \hat{u}(\boldsymbol{\xi}, \kappa_1, \boldsymbol{\eta}_{-1}) \times e^{2\pi i \langle x,\xi\rangle} \times e^{2\pi i \langle \boldsymbol{\kappa}_{-1}, \boldsymbol{\eta}_{-1}\rangle} d\boldsymbol{\xi}\, d\boldsymbol{\eta}_{-1}.$$

Craig & Weinstein (2008) | Wang et al. (2022) | Dinov & Velev (2021)

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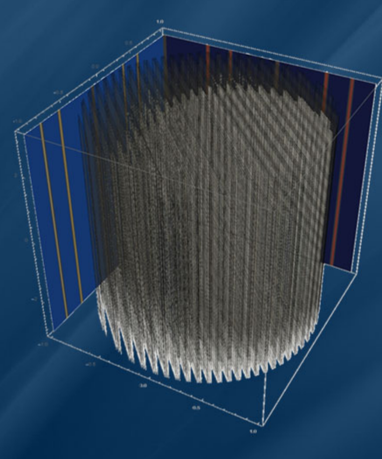
A *Spacekime* Solution to Wave Equation

- Math Generalizations: Derived <u>other spacekime concepts</u>: law of addition of velocities, energy-momentum conservation law, stability conditions for particles moving in spacekime, conditions for nonzero rest particle mass, causal structure of spacekime, and solutions of the ultrahyperbolic wave equation under Cauchy initial data …



Wang et al., 2022 | Dinov & Velev (2021)

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Math Foundations of Spacekime

- **Spacekime:** $(\boldsymbol{x}, \boldsymbol{k}) = \left(\underbrace{x^1, x^2, x^3}_{\text{space}}, \underbrace{c\kappa_1 = x^4, c\kappa_2 = x^5}_{\text{kime}} \right) \in X$
- **Kevents** (*complex events*): points (or states) in the spacekime manifold X. Each kevent is defined by where ($\boldsymbol{x} = (x, y, z)$) it occurs in space, what is its *causal longitudinal order* $\left(r = \sqrt{(x^4)^2 + (x^5)^2} \right)$, and in what *kime-direction* ($\varphi = \operatorname{atan2}(x^5, x^4)$) it takes place
- **Spacekime interval** (ds) is defined using the general Minkowski 5×5 metric tensor
- **Spacekime Calculus of differentiation and integration** (defined using Wirtinger derivatives and path integration
- Generalization of the **equations of motion in spacekime**
- **Lorentz transformation** (between 2 spacekime inertial frames)
- Solutions to ultrahyperbolic PDEs

Dinov & Velev (2021)

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Spacekime Calculus

- Kime **Wirtinger derivative**, 1st order kime-derivative at $\boldsymbol{k} = (r, \varphi)$, $z = (x + iy)$:

$$f'(z) = \frac{\partial f(z)}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \text{ and } f'(\bar{z}) = \frac{\partial f(\bar{z})}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

In Conjugate-pair basis: $df = \partial f + \bar{\partial} f = \frac{\partial f}{\partial z} dz + \frac{\partial}{\partial \bar{z}} d\bar{z}$

In Polar kime coordinates:

$$f'(k) = \frac{\partial f(k)}{\partial k} = \frac{1}{2}\left(\cos\varphi \frac{\partial f}{\partial r} - \frac{1}{r}\sin\varphi \frac{\partial f}{\partial \varphi} - i\left(\sin\varphi \frac{\partial f}{\partial r} + \frac{1}{r}\cos\varphi \frac{\partial f}{\partial \varphi}\right)\right) = \frac{e^{-i\varphi}}{2}\left(\frac{\partial f}{\partial r} - \frac{i}{r}\frac{\partial f}{\partial \varphi}\right)$$

$$f'(\bar{k}) = \frac{\partial f(\bar{k})}{\partial \bar{k}} = \frac{1}{2}\left(\cos\varphi \frac{\partial f}{\partial r} - \frac{1}{r}\sin\varphi \frac{\partial f}{\partial \varphi} + i\left(\sin\varphi \frac{\partial f}{\partial r} + \frac{1}{r}\cos\varphi \frac{\partial f}{\partial \varphi}\right)\right) = \frac{e^{i\varphi}}{2}\left(\frac{\partial f}{\partial r} + \frac{i}{r}\frac{\partial f}{\partial \varphi}\right).$$

- Kime **Wirtinger integration**:

Path-integral $\lim_{|z_{m+1} - z_m| \to 0} \sum_{m=1}^{n-1} (f(z_m)(z_{m+1} - z_m)) \cong \oint_{z_a}^{z_b} f(z) dz$.

Definite area integral: for $\Omega \subseteq \mathbb{C}$, $\int_\Omega f(z) dz d\bar{z}$.

Indefinite integral: $\int f(z) dz d\bar{z}$, $df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$.

The *Laplacian* in terms of conjugate pair coordinates is $\Delta f = d^2 f = 4 \frac{\partial f}{dz}\frac{\partial f}{d\bar{z}} = 4 \frac{\partial f}{d\bar{z}}\frac{\partial f}{dz}$.

Dinov & Velev (2021)

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(Many) Spacekime Open Math Problems

- **Analyticity** – study the holomorphic properties of the data/signals in Spacekime
  - Investigate the relation between time → kime transformations $\mathcal{L} = \{t \in \mathbb{R} \to \kappa \in \mathbb{C}\}$ and the analytical properties of the resulting kimesurfaces $\left(\tilde{f}(\kappa): \mathbb{C} \to \mathbb{C}\right)$ corresponding to the originally observed time-series processes $(f(t): \mathbb{R}^+ \to \mathbb{R}, \mathbb{C})$.
  - This knowledge may enhance our understanding of, and potentially suggest novel, AI/ML/statistical/data-science methods for modeling, prediction, inference or forecasting on observed longitudinal data.
- **Ergodicity**: Look at particle velocities in the 4D Minkowski spacetime (X), a measure space where gas particles move spatially and evolve longitudinally in time. Let $\mu = \mu_x$ be a measure on X, $f(\boldsymbol{x}, t) \in L^1(X, \mu)$ be an integrable function (e.g., velocity of a particle), and $T: X \to X$ be a measure-preserving transformation at position $\boldsymbol{x} \in \mathbb{R}^3$ and time $t \in \mathbb{R}^+$.
  - A pointwise ergodic theorem argues that in a measure theoretic sense, the average of f (e.g., velocity) over all particles in the gas system at a fixed time, $\bar{f} = \mathbb{E}_t(f) = \int_{\mathbb{R}^3} f(\boldsymbol{x}, t) d\mu_x$, is equal to the average f of just one particle ($\boldsymbol{x}$) over the entire time span,
  $$\tilde{f} \equiv \mathbb{E}_x(f) = \lim_{n \to \infty} \left(\frac{1}{n} \sum_{m=0}^{n} f(T^m \boldsymbol{x})\right), \text{ i.e., (show) } \bar{f} \equiv \tilde{f}.$$
  - μ_x is a prob measure and the transformation $T^m \boldsymbol{x}$ represents the time starting with an initial spatial location $T^o \boldsymbol{x} = \boldsymbol{x}$. Investigate the ergodic properties of various transform in 5D **spacekime**:

$$\underbrace{\bar{f} \equiv \mathbb{E}_\kappa(f) = \frac{1}{\mu_x(X)} \int f\left(\boldsymbol{x}, \underbrace{t, \phi}_{\kappa}\right) d\mu_x}_{\text{space averaging}} \stackrel{?}{=} \underbrace{\lim_{t \to \infty} \left(\frac{1}{t} \sum_{m=0}^{t} \left(\int_{-\pi}^{+\pi} f(T^m \boldsymbol{x}, t, \phi) d\Phi\right)\right) = \mathbb{E}_x(f) \equiv \tilde{f}}_{\text{kime averaging}}$$

Dinov & Velev (2021)

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Bayesian Inference Simulation

- Simulation example using 2 random samples drawn from mixture distributions each of $n_A = n_B = 10\text{K}$ observations
  - $\{X_{A,i}\}_{i=1}^{n_A}$, where $X_{A,i} = 0.3U_i + 0.7V_i$, $U_i \sim N(0,1)$ and $V_i \sim N(5,3)$, and
  - $\{X_{B,i}\}_{i=1}^{n_B}$, where $X_{B,i} = 0.4P_i + 0.6Q_i$, $P_i \sim N(20,20)$ and $Q_i \sim N(100,30)$.
- The intensities of cohorts A and B are independent and follow different mixture distributions. We'll split the first cohort (A) into training (C) and testing (D) subgroups, and then
  - Transform all four cohorts into Fourier k-space,
  - Iteratively randomly sample single observations from the (training) cohort C,
  - Reconstruct the data into spacetime using a single kime-magnitude value and alternative kime-phase estimates derived from cohorts B, C, and D, and
  - Compute the classical spacetime-derived population characteristics of cohort A and compare them to their spacekime counterparts obtained using a single C kime-magnitude paired with B, C, or D kime-phases.

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Bayesian Inference Representation

- Spacekime analytics based on a single spacetime observation x_{i_o} can be thought of as a type of Bayesian prior-predictive *or* posterior-predictive distribution estimation problem
  - Prior predictive distribution of a new data point x_{j_o}, marginalized over the *prior* – i.e., the sampling distribution $p(x_{j_o}|\gamma)$ weight-averaged by the pure *prior* distribution):

$$p(x_{j_o}|\varphi') = \int p(x_{j_o}|\gamma) \times \underbrace{p(\gamma|\,\varphi')}_{\text{prior distribution}}\, d\gamma\ .$$

  - Posterior predictive distribution of a new data point x_{j_o}, marginalized over the *posterior*; i.e., the sampling distribution $p(x_{j_o}|\gamma)$ weight-averaged by the *posterior* distribution:

$$p(x_{j_o}|x_{i_o},\varphi') = \int p(x_{j_o}|\gamma) \times \underbrace{p(\gamma|x_{i_o},\varphi')}_{\text{posterior distribution}}\, d\gamma\ .$$

- The difference between these two predictive distributions is that
  - the posterior predictive distribution is updated by the observation $X = \{x_{i_o}\}$ and the hyperparameter, φ (phase aggregator),
  - whereas the prior predictive distribution only relies on the values of the hyperparameters that appear in the prior distribution

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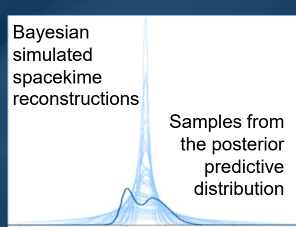
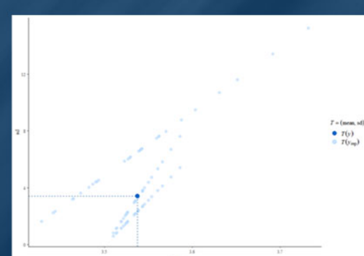
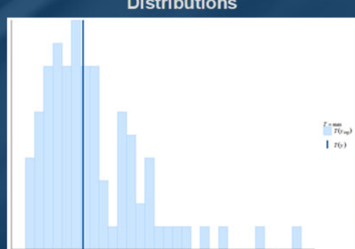
Bayesian Inference Simulation



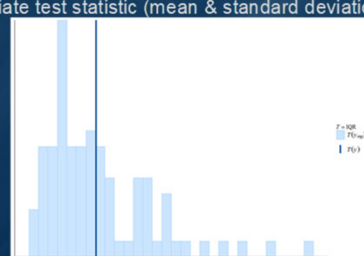
Distributions



Bivariate test statistic (mean & standard deviation)



Test statistic (maximum)



Test statistic (inter-quartile range, IQR)

Relations between the empirical data distribution (**dark blue**) and samples from the posterior predictive distribution, Bayesian simulated spacekime reconstructions (**light-blue**).

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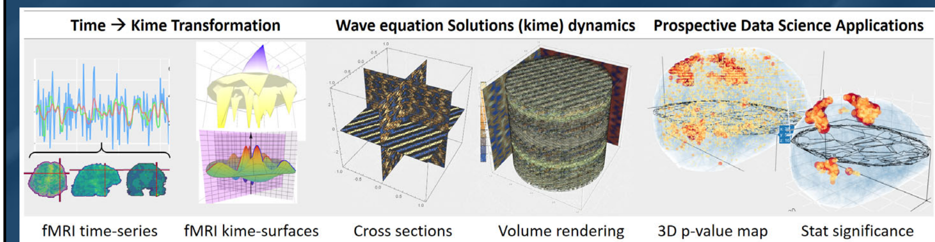
Data→ Kime-Transforms → PDEs → AI



Wang et al., 2022 | Dinov & Velev (2021)

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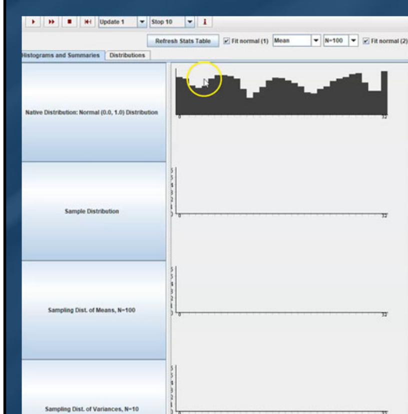
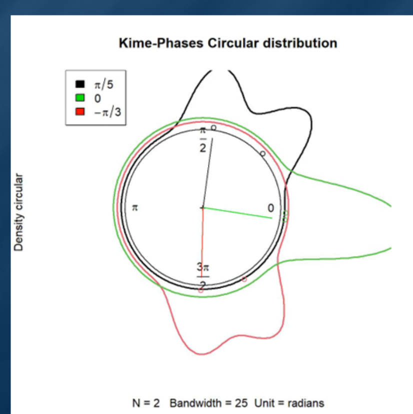
Random Sampling & Kime-Phase Paradigm

- ❑ Kime phase distributions are mostly symmetric → random observations ≡ phase sampling

https://wiki.socr.umich.edu/index.php/SOCR_EduMaterials_Activities_GeneralCentralLimitTheorem





https://www.socr.umich.edu/TCIU/HTMLs/Chapter6_Kime_Phases_Circular.html

Dinov, Christou & Sanchez (2008) Dinov & Velev (2021)

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Mathematical-Physics ⟹ Data Science & AI

Physics	Data/Neuro Sciences
A **particle** is a small localized object that permits observations and characterization of its physical or chemical properties	An **object** is something that exists by itself, actually or potentially, concretely or abstractly, physically or incorporeal (e.g., person, subject, etc.)
An **observable** a dynamic variable about particles that can be measured	A **feature** is a dynamic variable or an attribute about an object that can be measured
Particle **state** is an observable particle characteristic (e.g., position, momentum)	**Datum** is an observed quantitative or qualitative value, an instantiation, of a feature
Particle **system** is a collection of independent particles and observable characteristics, in a closed system	**Problem**, aka *Computable Data Object*, is a collection of independent objects and features, without necessarily being associated with a priori hypotheses
Wave-function	**Inference-function**
Reference-Frame **transforms** (e.g., Lorentz)	Data **transformations** (e.g., wrangling, log-transform)
State of a system is an observed measurement of all particles ~ wavefunction	**Dataset (data)** is an observed instance of a set of datum elements about the problem system, $\boldsymbol{O} = \{\boldsymbol{X}, \boldsymbol{Y}\}$
A **particle system is computable** if (1) the entire system is logical, consistent, complete and (2) the unknown internal states of the system don't influence the computation (wavefunction, intervals, probabilities, etc.)	**Computable data object** is a very special representation of a dataset which allows direct application of computational processing, modeling, analytics, or inference based on the observed dataset
…	…

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Mathematical-Physics ⟹ Data Science & AI

Physics

Wavefunction

Wave equ problem:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t}\right)\psi(x,t) = 0$$

Complex Solution:
$\psi(x,t) = Ae^{i(kx-wt)}$
represents a traveling wave,
where $\left|\frac{w}{k}\right| = v$.

Data Science

Inference function - describing a solution to a specific data analytic system (a problem). For example,

- A linear (GLM) model represents a solution of a prediction inference problem, $\boldsymbol{Y} = \boldsymbol{X}\beta$, where the inference function quantifies the effects of all independent features ($\boldsymbol{X}$) on the dependent outcome ($\boldsymbol{Y}$), data: $\boldsymbol{O} = \{\boldsymbol{X}, \boldsymbol{Y}\}$:
$$\psi(\boldsymbol{O}) = \psi(\boldsymbol{X}, \boldsymbol{Y}) \;\;\Rightarrow\;\; \hat{\beta} = \hat{\beta}^{OLS} = \langle X|X\rangle^{-1}\langle X|Y\rangle = \left(X^T X\right)^{-1} X^T Y.$$
- A non-parametric, non-linear, alternative inference is SVM classification. If $\psi_x \in \boldsymbol{H}$, is the lifting function $\psi: \boldsymbol{R}^\eta \to \boldsymbol{R}^d$ ($\psi: x \in \boldsymbol{R}^\eta \to \tilde{x} = \psi_x \in \boldsymbol{H}$), where $\eta \ll d$, the kernel $\psi_x(y) = \langle x|y\rangle: \boldsymbol{O} \times \boldsymbol{O} \to \boldsymbol{R}$ transformes non-linear to linear separation, the observed data $\boldsymbol{O}_i = \{x_i, y_i\} \in \boldsymbol{R}^\eta$ are lifted to $\psi_{O_i} \in \boldsymbol{H}$. The SVM prediction operator is the weighted sum of the kernel functions at ψ_{O_i}, where β^* is a solution to the SVM regularized optimization:
$$\underbrace{\langle \psi_O | \beta^* \rangle_H}_{predictions} = w^T x + b = \sum_{i=1}^n p_i^* \langle \psi_O | \psi_{O_i} \rangle_H + b,$$
$$\min_{w \in \mathbb{R}^d,\, \xi \in \mathbb{R}^+} \left(\overbrace{\|w\|^2}^{regularizer} + C \overbrace{\sum_{i=1}^m \xi_i}^{fidelity} \right), y^{(i)}\left(w^T x^{(i)} + b\right) \geq 1 - \xi_i, \xi_i \geq 0$$

The dual weight coefficients, p_i^*, are multiplied by the label corresponding to each training instance, $\{y^{(i)}\}$.

Inference always depends on the (input) data; however, it does not have 1-1 and onto bijective correspondence with the data, since the inference function quantifies predictions in a probabilistic sense.

GLM/SVM: https://DSPA2.predictive.space | Dinov, Springer (2018, 2023)

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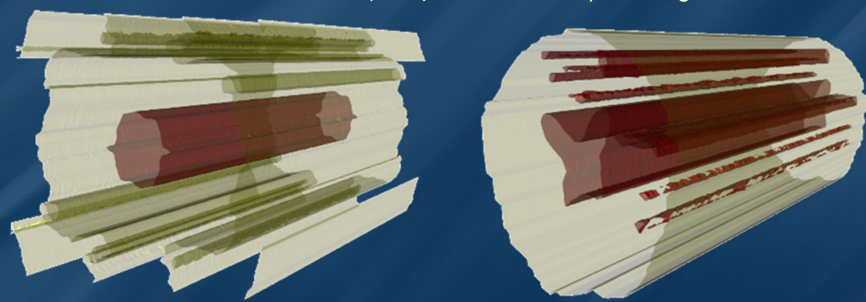
Spacekime Analytics: fMRI Example

- 3D Isosurface Reconstruction of (2D *space* × 1D *time*) fMRI signal



<u>Spacetime</u> **Reconstruction using trivial phase-angle; kime=time=(magnitude, 0)**

<u>Spacekime</u> **Reconstruction using correct kime=(magnitude, phase)**

3D pseudo-spacetime reconstruction:

$$f = \hat{h}\,(\underbrace{x_1, x_2}_{space},\ \underbrace{t}_{time})$$

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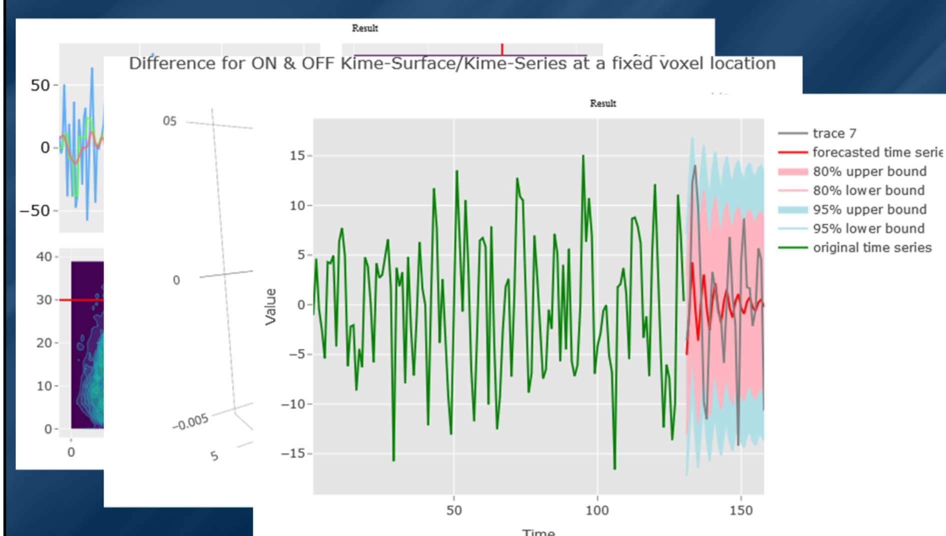
Spacetime Time-series ⟹ *Spacekime Kimesurfaces* ⟹ *TLM*



Zhang et al., 2022 | Dinov & Velev (2021)

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Mapping Longitudinal Data (Time-series) to Kime-Surfaces

The *forward* and *inverse* (continuous) Laplace transforms are defined below.

- For a given function (of time) $f(t): \mathbb{R}^+ \to \mathbb{C}$, the **Laplace transform** is the function of a complex frequency argument, $F(z) = \mathcal{L}(f)(z): \mathbb{C} \to \mathbb{C}$:

$$\mathcal{L}(f)(z) = F(z) = \int_0^\infty f(t)e^{-zt}dt\,.$$

- For a given function of a complex frequency argument, $F(z)$, the **Inverse Laplace transform** (ILT) is the function of a positive real (time-like) argument $f(t) = \mathcal{L}^{-1}(F)(t): \mathbb{R}^+ \to \mathbb{C}$, which is defined in terms of a complex path integral (a.k.a. Bromwich integral or Fourier–Mellin integral):

$$f(t) = \mathcal{L}^{-1}(F)(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{zt}F(z)dz\,,$$

where the parameter $\gamma \in \mathbb{R}$ is chosen so that the entire complex contour path of the integral is inside of the region of convergence of $F(z)$.

In probability and statistics, the Laplace transform plays the role of expected value. If X is a random variable, then its Laplace transform, i.e., the LT of its probability density function f_X, is given by the expectation of an exponential: $\mathcal{L}(X) = \mathcal{L}(f)(z) = \mathbb{E}(e^{-zX})$.

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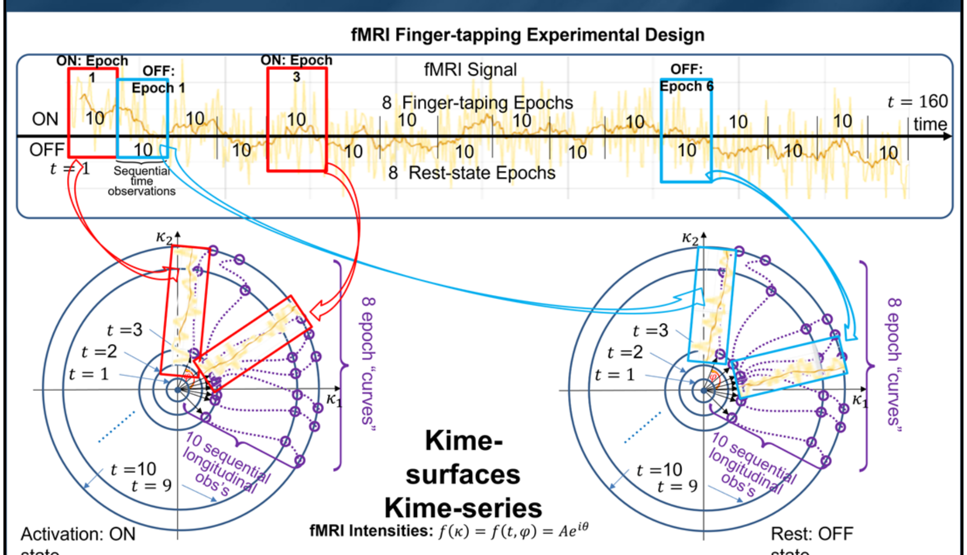
Mapping Longitudinal Data (Time-series) to Kime-Surfaces



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Mapping Longitudinal Data (Time-series) to Kime-Surfaces

Apply the ILT ($\mathcal{L}^{-1}$) to reconstruct a time-series, $\hat{f}(t) = \mathcal{L}^{-1}(F)(t)$:

$$F(z) = \mathcal{L}(f) = \underbrace{\frac{1}{z+1}}_{F_1(z)=\mathcal{L}(f_1(t)=e^{-t})} + \underbrace{\frac{1}{z^2+1}}_{F_2(z)=\mathcal{L}(f_2(t)=\sin(t))} \times \underbrace{\frac{z}{z^2+1}}_{F_3(z)=\mathcal{L}(f_3(t)=\cos(t))} + \underbrace{\frac{1}{z^2}}_{F_4(z)=\mathcal{L}(f_4(t)=t)},$$

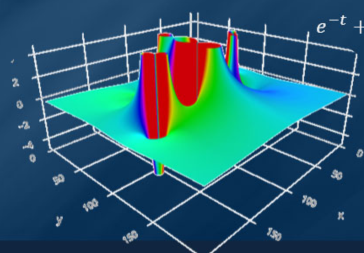
$$f(t) = \mathcal{L}^{-1}(F) = \mathcal{L}^{-1}(F_1 + F_2 \times F_3 + F_4)) =$$

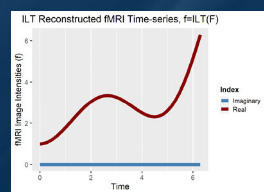
$$\mathcal{L}^{-1}(F_1) + \left(\underbrace{\mathcal{L}^{-1}(F_2) * \mathcal{L}^{-1}(F_3)}_{\text{convolution}}\right)(t) + \mathcal{L}^{-1}(F_4) =$$

$$\mathcal{L}^{-1}(\mathcal{L}(f_1))(t) + \left(\mathcal{L}^{-1}(\mathcal{L}(f_2)) * \mathcal{L}^{-1}(\mathcal{L}(f_3))\right)(t) + \mathcal{L}^{-1}(\mathcal{L}(f_4))(t),$$

$$f(t) = \mathcal{L}^{-1}(F)(t) = f_1(t) + (f_2 * f_3)(t) + f_4(t) =$$

$$e^{-t} + \int_0^t \sin(\tau) \times \cos(t-\tau)d\tau + t = t + e^{-t} + \frac{t\sin(t)}{2}.$$





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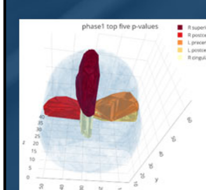
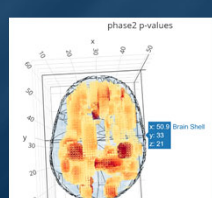
Tensor-based Linear Modeling of fMRI

3-Step Analysis: registering the fMRI data into a brain atlas space, 56 ROIs, tensor linear modeling, post-hoc FDR processing & selection of large clusters of significant voxels are identified within the important ROIs: $Y = \underbrace{\langle X, B\rangle}_{\text{tensor product}} + E$.

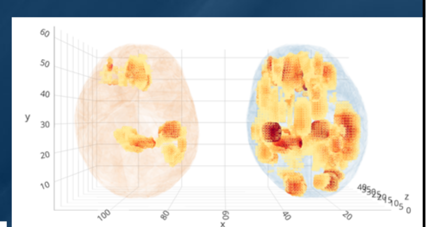
The dimensions of the time-tensor Y are $\overbrace{160}^{\text{time}} \times \overbrace{a \times b \times c}^{\text{ROI b-box}}$, where the tensor elements represent the response variable $Y[t,x,y,z]$, i.e., fMRI intensity. For fMRI magnitude (real-valued signal), the design kime-tensor X dimensions are: $\underbrace{10 * 8}_{\text{Kime}(\text{Time}*e^{i\times Repeat})} \times \underbrace{State}_{\text{Stim vs. Rest (2)}} \times \underbrace{4}_{\text{effects}} \times \underbrace{1}_{\mathbb{R}}$.



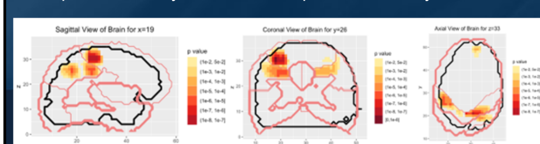
Step 1: ROI analysis



Step 2: Voxel analysis



Voxel-based TLM/Analysis
FDR Corrected (step 3, left) vs. Raw (step 2, right)



Step 3: 2D voxel analysis projections (finger-tapping task modeling)

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Spacekime Analytics: Resources & Demos

- Tutorials
  - https://TCIU.predictive.space
  - https://SpaceKime.org
- R Package
  - https://cran.rstudio.com/web/packages/TCIU
- GitHub
  - https://github.com/SOCR/TCIU
- Pubs
  - https://socr.umich.edu/people/dinov/publications.html

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